Chapter 5

Power Series

5.1 Infinite Series of Complex Numbers

Definition 5.1.1

An infinite series

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots + a_n + a_{n+1} + \dots$$

(with $a_k \in \mathbb{C}$) converges to S if the sequence of partial sums $\{S_n\}$, given by

$$S_n = \sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots + a_n$$

converges to S.

- a_n is an expression that gives the nth term in a sequence.
- S_n is an expression that gives the sum of the first n terms.

Example.

Consider
$$\sum_{k=0}^{\infty} z^k$$
for some $z \in \mathbb{C}$. We have that

$$S_n = 1 + z + z^2 + \cdots + z_n$$

Can we find a closed formula for S_n, to help us find the limit as $n \to \infty$?

$$\begin{aligned} S_n &= 1 + z + z^2 + \cdots + z^n \\ z \cdot S_n &= z + z^2 + z^3 + \cdots + z^{n+1}, \text{ thus} \\ S_n - zS_n &= 1 - z^{n+1} \end{aligned}$$

Hence $S_n = \frac{1-z^{n+1}}{1-z}$, for $z \neq 1$, and since $z^{n+1} \to 0$ as $n \to \infty$ as long as $|z| < 1$ we have that
$$\sum_{k=0}^{\infty} \frac{1}{1-z} \quad , |z| < 1.$$

Example.

What is $\displaystyle\sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n$?

$$\begin{aligned} S_n &= 1 + \frac{i}{3} + \left(\frac{i}{3}\right)^2 + \cdots + \left(\frac{i}{3}\right)^n \\ \frac{i}{3} \cdot S_n &= \left(\frac{i}{3}\right) + \left(\frac{i}{3}\right)^2 + \left(\frac{i}{3}\right)^3 + \cdots + \left(\frac{i}{3}\right)^{n+1}, \text{ thus} \\ S_n - \left(\frac{i}{3}\right)S_n &= 1 - \left(\frac{i}{3}\right)^{n+1} \\ S_n &= \frac{1 - (\frac{i}{3})^{n+1}}{1 - \frac{i}{3}} \end{aligned}$$

Since $|\frac{i}{3}| < 1$, the formula we just found implies

$$\sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n = \frac{1}{1-\frac{i}{3}} = \frac{1}{\frac{3-i}{3}} = \frac{3}{3-i} = \frac{3(3+i)}{(3-i)(3+i)} = \frac{9+3i}{10}$$

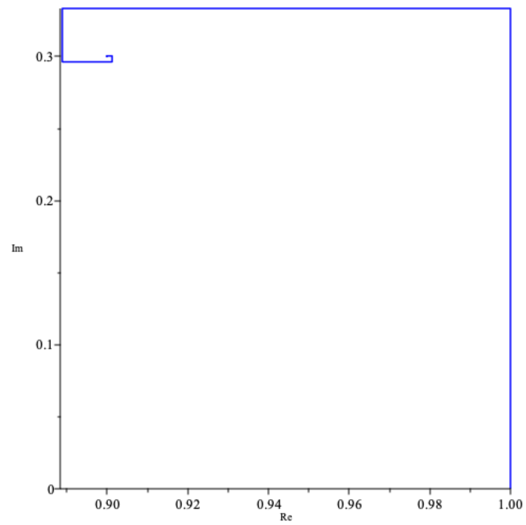

Figure 5.1: Visualization of the convergence behavior of the partial sums of the geometric series $\sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n$. The plot shows the sequence approaching the limit $\frac{9}{10} + \frac{3}{10}i$ in the complex plane, exhibiting a spiraling path characteristic of complex geometric series with nonzero real and imaginary parts.

What happens for $|z| > 1$?

Theorem 5.1.2

If a series $\sum_{k=0}^{\infty}$ converges then $a_k \to 0$ as $k \to \infty$.

In the previous example, if $|z| \geq 1$, then $|z|^k \not\to 0$ as $k \to \infty$, thus $\sum_{k=0}^{\infty} z^k$ does not converge for $|z| \geq 1$. We say the series diverges for $|z| \geq 1$

Let's now analyze the real and imaginary parts of the equation $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for $|z| < 1$:

Let's write z in polar form. Writing $z = re^{i\theta}$ we have $z^k = r^k e^{ik\theta} = r^k \cos(k\theta) + r^k i \sin(k\theta)$. Thus

$$\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} r^k \cos(k\theta) + i \sum_{k=0}^{\infty} r^k \sin(k\theta)$$

Furthermore,

$$\begin{aligned}\frac{1}{1-z} = \frac{1}{1-re^{i\theta}} &= \frac{1-re^{-i\theta}}{(1-re^{i\theta})(1-re^{-i\theta})} \\ &= \frac{1-r\cos\theta + ir\sin\theta}{1-re^{-i\theta}-re^{i\theta}+r^2} = \frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2}\end{aligned}$$

Thus

$$\sum_{k=0}^{\infty} r^k \cos(k\theta) = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} \text{ and } \sum_{k=0}^{\infty} r^k \sin(k\theta) = \sum_{k=0}^{\infty} \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

Example.

What is $\displaystyle\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$?

$$\begin{aligned} S_n &= 1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 + \cdots + \left(\frac{i}{2}\right)^n \\ \frac{i}{2} \cdot S_n &= \left(\frac{i}{2}\right) + \left(\frac{i}{2}\right)^2 + \left(\frac{i}{2}\right)^3 + \cdots + \left(\frac{i}{2}\right)^{n+1}, \text{ thus} \\ S_n - \left(\frac{i}{2}\right) S_n &= 1 - \left(\frac{i}{2}\right)^{n+1} \\ S_n &= \frac{1 - (\frac{i}{2})^{n+1}}{1 - \frac{i}{2}} \end{aligned}$$

Since $|\frac{i}{2}| < 1$, the formula we just found implies

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = \frac{1}{1 - \frac{i}{2}} = \frac{1}{\frac{2-i}{2}} = \frac{2}{2-i} = \frac{2(2+i)}{(2-i)(2+i)} = \frac{4+2i}{5}$$

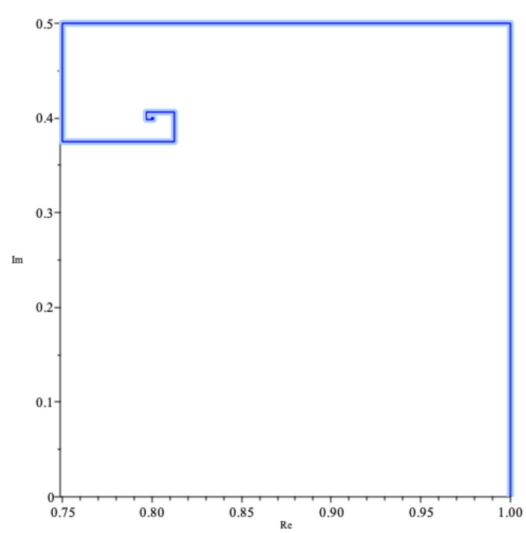

Figure 5.2: Visualization of the convergence behavior of the partial sums of the geometric series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$.

Example.

We just saw that $\sum_{n-0}^{\infty}\left(\frac{i}{2}\right)^n = \frac{4+2i}{5}$. Use it to find $1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \frac{1}{2^8} - \ldots$.

We know that if n is even (that is, n is of the form $n = 2k$), then $i^n = i^{2k} = (-1)^k$; if n is odd (that is, n is of the form $n = 2k+1$), then $i^n = i^{2k+1} = i(-1)^k$.

Now,

$$\begin{aligned}\frac{4+2i}{5} = \sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^n &= \sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^{2k} + \sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^{2k+1} \\ &= \sum_{k=0}^{\infty}\frac{i^{2k}}{2^{2k}} + \sum_{k=0}^{\infty}\frac{i^{2k+1}}{2^{2k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}} + i\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+1}}\end{aligned}$$

Two complex numbers are equal if and only if their real and imaginary parts agree, so, looking at the real parts of the above only, it follows that

$$1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \frac{1}{2^8} - \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}} = \frac{4}{5}$$

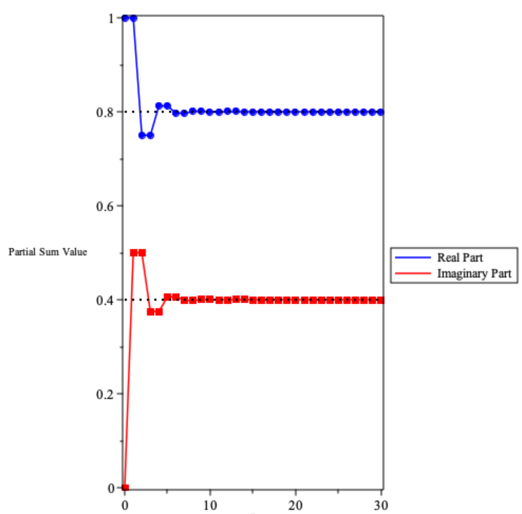

Figure 5.3: Visualization of Real and Imaginary Parts of Partial Sums of the geometric series $\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n$.

Example.

Consider $\sum_{k=1}^{\infty} \frac{i^k}{k}$. Does this series converge?

We note that $\sum_{k=1}^{\infty} \left| \frac{i^k}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series, which is know to diverge. One way to see this:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{\geq \frac{1}{2}} + \cdots$$

But does the series itself (without the absolute values) converge?

Let's split it up into the real and imaginary parts.

Note: When k is even (i.e k is of the form $k = 2n$, then $i^k = i^{2n} = (-1)^n$ is real. When k is odd (i.e k is of the form $k = 2n + 1$), then $i^k = i^{2n+1} = i(-1)^n$ is purely imaginary. Thus

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{i^k}{k} &= \sum_{n=1}^{\infty} \frac{i^{2n}}{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{2n+1} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \end{aligned}$$

is the alternating harmonic series, which converges.

But

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$

A similar argument leads to the imaginary part converging as well.

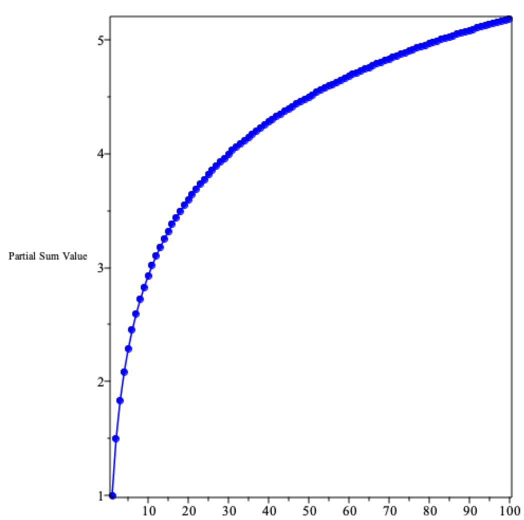

Figure 5.4: Growth of Harmonic Series Partial Sums

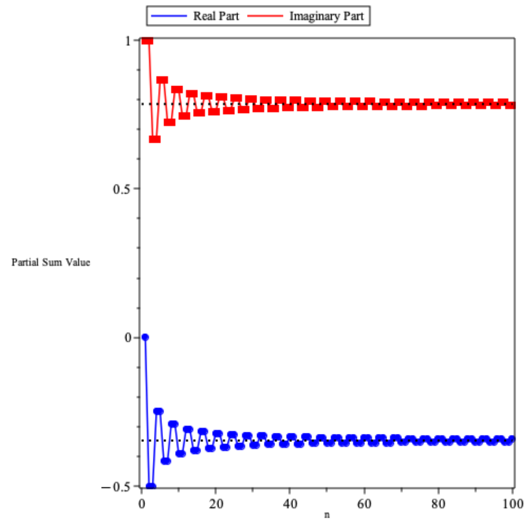

Figure 5.5: Real and Imaginary Parts of Partial Sums of $\sum_{k=1}^{\infty} \frac{i^k}{k}$

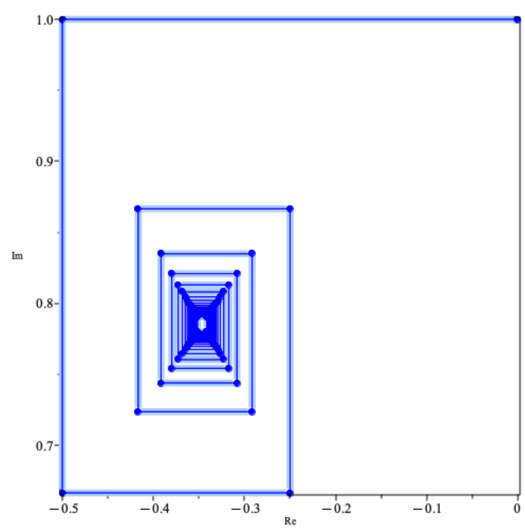

Figure 5.6: "Partial Sums of $\sum_{k=1}^{\infty} \frac{i^k}{k}$ in the Complex Plane

Example.

- $\sum_{n=0}^{\infty}\left(-\frac{2}{3}i\right)^n$. Since $\left|-\frac{2}{3}i\right| < 1$, the series converges.
- $\sum_{n=0}^{\infty}(\pi i)^n$. Since $|\pi i| = |\pi| > 1$, the common ratio has modulus greater than 1, so by the divergence criterion for geometric series, the series diverges.
- $\sum_{n=1}^{\infty}\frac{e^{\pi n i}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is the alternating harmonic series and it therefore converges.
- $\sum_{n=0}^{\infty}e^{2\pi n i} = \sum_{n=0}^{\infty}(-1)^{2n} = \sum_{n=0}^{\infty}(1)^n$, clearly diverges since it fails the necessary convergence criterion that the terms of the series have to go to zero.
- $\sum_{n=1}^{\infty}\frac{2i^n}{n}$, this series converges.

Absolute Convergence

Definition 5.1.3

A series $\sum_{k=0}^{\infty}a_k$ converges absolutely if the series $\sum_{k=0}^{\infty}|a_k|$ converges.

Example.

$\sum_{k=0}^{\infty}z^k$ converges and converges absolutely for $|z| < 1$

Example.

$\sum_{k=1}^{\infty}\frac{i^k}{k}$ converges, but not absolutely

Theorem 5.1.4

If $\sum_{k=0}^{\infty}a_k$ converges absolutely, then it also converges, and $\left|\sum_{k=0}^{\infty}a_k\right| < \sum_{k=0}^{\infty}|a_k|$