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Gravitational Stability of the Interface in Water Over Steam Geothermal Reservoirs

GEORGE TSYPKIN[1] and ANDREJ IL'ICHEV[2]
[1]*Institute for Problems in Mechanics, RAS, Av. Vernadskogo 101, 119420 Moscow, Russia*
[2]*Steklov Mathematical Institute, RAS, Gubkina Str. 8, 119991 Moscow, Russia*

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Abstract. Stability of a geothermal system is considered in a case when the water layer lies over the layer of superheated vapor in a stratum having relatively low permeability. This stratum locates between two parallel high permeable layers. Under the assumption of smallness of advective energy transfer as compared with the conductive one, the stationary distribution of the characteristics in the stratum with an interface of phase transition is obtained. The interface separates the domains occupied by water and vapor. Investigation of normal stability of the interface shows, that stable configurations in the geothermal system under consideration exist within the range of permeability values bounded by $k \sim 0.6 \times 10^{-15}\,\mathrm{m}^2$ from above. The most unstable configurations occur to be the quiescent states when the permeability exceeds a certain threshold. A sufficiently high value of permeability, satisfying the criterion of smallness of the advective energy transfer as compared with the conductive one makes it possible to explain the existence of a wide class of stable natural geothermal reservoirs, where the vapor layer underlies the water one.

Key words: geothermal reservoir, stationary solution, interface, normal stability.

Nomenclature

a	thermal diffusivity [$\mathrm{m}^2\,\mathrm{s}^{-1}$].
C	specific heat at constant pressure [$\mathrm{J\,K}^{-1}\,\mathrm{kg}^{-1}$].
g	acceleration of gravity [$\mathrm{m\,s}^{-2}$].
h	location parameter of the interface [m].
k	permeability [m^2].
L	thickness of the low permeable stratum [m].
m	porosity [1].
P	pressure [Pa].
q	specific heat of phase transition [$\mathrm{J\,kg}^{-1}$].
t	time [s].
T	temperature [K].
V	speed of the phase transition interface [$\mathrm{m\,s}^{-1}$].
x	vertical coordinate [m].

Greek Symbols

η	perturbation of the interface [m].
κ	wave number [m^{-1}].
λ	thermal conductivity [$\mathrm{W\,m}^{-1}\,\mathrm{K}$].

μ	viscosity [Pa s].
ρ	density [kg m^{-3}].
σ	spectral parameter [s^{-1}].

Subscripts

1	water domain.
2	vapor domain.
n	normal.
s	skeleton of porous medium.
v	vapor.
w	water.
0	boundary value at $x = 0$.
$+$	right ahead of the interface in the water saturated domain.
$-$	right behind the interface in the vapor domain.
$*$	at the phase transition front.

Superscript

0	boundary value at $x = L$.

1. Introduction

Full-scale investigations of natural geothermal systems showed, that in great number of reservoirs the situation takes place, when a water layer of a considerable thickness locates over a layer of superheated vapor (White *et al.*, 1971; Grant, 1983). The existence of such a configuration is explained from the thermodynamic point of view by a temperature gradient which presence in geothermal systems is typical. In this case the thermodynamic conditions at large depths relate to the domain occupied by vapor, and at smaller depths – to the domain occupied by water. On the other hand, investigation of hydrodynamic stability shows, that the state, when the layer of heavy liquid lies over the layer of lighter one occurs to be unstable if the layers are immiscible (Chandrasekhar, 1961). In order to explain the possibility of existence of stable geothermal systems containing the water layer overlying the vapor layer, some qualitative hypotheses about physical mechanisms of stability of such a configuration were proposed (Grant, 1983).

In Schubert and Straus (1980), the example of the geothermal system is given, where the water layer overlies the vapor one, and also the stability analysis of the phase transition interface, separating water and vapor domains, was fulfilled. It was assumed, that the lowest boundary of the stratum is a contact surface of non-permeable rock and the vapor saturated permeable domain. It was also assumed that in the unperturbed state the phases are motionless and the phase transition is absent. Numerical analysis of the dispersion equation implies that the configuration in question may be both stable and unstable. The critical value of the permeability coefficient separating the domain of stable and unstable states of the system was found to be given by $k \sim 4 \times 10^{-17}$ m^2. For values of permeability higher than the critical one the system loses its stability. It was noted that the found critical value of permeability is less by the order of magnitude than the value characteristic for natural geothermal systems, and thus, the stability of most systems cannot be explained.

In the present paper we propose a more complex example of a geothermal system where motion of phases and the phase transition in the unperturbed state are taken into account. A solution of the steady-state bounded problem with a water–vapor phase transition interface is obtained under the assumption of smallness of the advective energy transfer as compared with the conductive one. There are two different solutions: the first one corresponds to vaporization and describes water motion towards the interface, the other one corresponds to vapor condensation and describes vapor flow in the direction of the interface. The quantities to be determined are the location of the phase transition interface and also the vaporization temperature and pressure on this surface. These quantities are found from the solving of the transcendental equation on the interface. The calculations show that at low values of permeability the transcendental equation has a unique solution, which determines the location of the interface of phase transitions. At comparatively high values of permeability of order $k \sim 10^{-16}\,\mathrm{m}^2$ and higher, the transcendental equation in question has the additional two roots. It means that for the same values of characteristics at the boundaries, there exist three different locations of the interface.

The topological method was used for the investigation of normal stability of the interface, based on the argument principle. This method makes it possible to detect complex roots of the dispersion equation, having the positive real part. The presence of these roots corresponds to the instability of the solution under consideration. It is essential, that we give no assumptions about the real nature of the roots of the dispersion equation. This assumption is contained implicitly in Schubert and Straus (1980). Our topological approach also allows to determine the number of the roots of the dispersion equation.

Investigation of stability of the interface shows that at values of permeability $k \sim 10^{-16}\,\mathrm{m}^2$ and lower, that is, when there exists the unique solution of the stationary problem, the interface is always stable. In the case when there exist three solutions of the stationary problem, the configuration, corresponding to the dynamic equilibrium without phase transition when water and vapor are motionless, is always unstable. The second solution, relating to the water motion downwards and its further vaporization, is unstable with respect to short wavelength perturbations. Numerical investigation shows that the third solution, which describes the process of vapor motion upwards and its condensation at the interface, can be stable also at comparatively high values of permeability $k \sim 6 \times 10^{-16}\,\mathrm{m}^2$.

It seems to be important to note that the regimes obtained are realized within the range of parameters which appear to be characteristic for the natural geothermal systems. Stability mechanism of the class of geothermal systems under consideration has the clear physical meaning, implying the predominance of the conductive energy transfer over the advective one.

The paper is organized as follows. In Section 2 the mathematical model of phase transition problem for the bounded domain is formulated. Neglecting advective heat transfer, we simplify the basic system in order to apply normal mode stability analysis directly. In Section 3 we derive and investigate the steady-state solutions for the phase transition problem. In Section 4 we present the linear stability analysis and show that the water over steam configuration can exist for high permeability geothermal reservoirs. In Section 5 we discuss the range of applicability of our analysis and the conditions when a stable water over steam configuration may arise.

2. Formulation of the Problem

We consider a high-temperature geothermal reservoir, consisting of two high permeability layers, which are separated by a low permeability stratum. Let us assume that the thermodynamic conditions imply that the upper high permeability layer is filled in by water and the lower one – by vapor. Then in the low permeability layer there exists the phase transition interface, separating domains occupied by water and vapor, respectively (Figure 1).

In dependence on values of pressure in the high permeability layers, either the regime of vaporization, when the water moves downwards, or the regime of condensation, corresponding to the vapor motion upwards, take place. In the rest state phase transitions are absent, and the pressure distribution in the high permeability layers coincides with the hydrostatic one.

Processes of heat and mass transfer in the framework of equilibrium thermodynamics can be described by mass and energy conservation laws, the Darcy law for water and vapor with allowance for gravity, the equations of state and thermodynamic relations (O'Sullivan, 1985). Following Chandrasekhar (1961) and Schubert

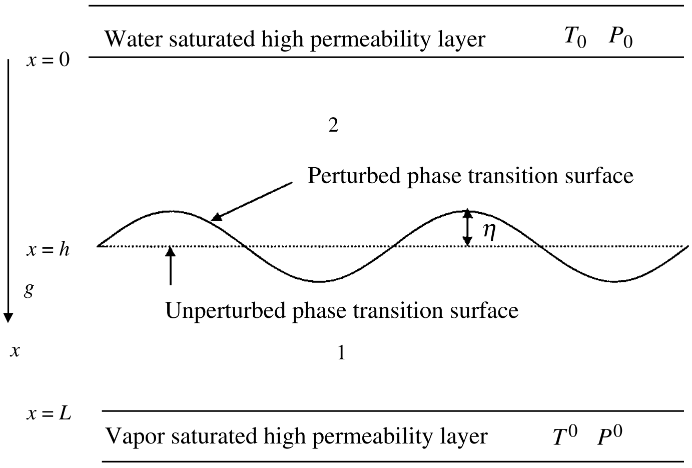

Figure 1. Sketch of the problem (1) vapor domain, and (2) water domain in the low permeability stratum.

and Straus (1980), we assume that the water and vapor are incompressible. Then the basic system of equations for two domains of the low permeability stratum separated by the interface is written as

$$\begin{aligned}
&\operatorname{div} v_j = 0, \quad v_j = -\frac{k}{\mu_j}\,(\operatorname{grad} P - \rho_j g \mathbf{e}_x),\\
&(\rho C)_{1,2}\frac{\partial T}{\partial t} + \rho_j C_j v_j \operatorname{grad} T = \operatorname{div}(\lambda_{1,2}\operatorname{grad} T),\\
&\lambda_{1,2} = m\lambda_j + (1-m)\lambda_{\mathrm{s}},\\
&(\rho C)_{1,2} = m\rho_j C_j + (1-m)\rho_{\mathrm{s}} C_{\mathrm{s}}, \quad j = \mathrm{v}, \mathrm{w}. \qquad (1)
\end{aligned}$$

The conservation of mass and energy across the interface are formulated as the conditions of the thermodynamic equilibrium jump of the water saturation function (Fitzgerald and Woods, 1994; Tsypkin, 1994, 1997). These relations have the form

$$\begin{aligned}
&m\left(1-\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{w}}}\right)V_{\mathrm{n}} = \frac{k}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{w}}}(\operatorname{grad} P)_{\mathrm{n}+} - \frac{k}{\mu_{\mathrm{w}}}(\operatorname{grad} P)_{\mathrm{n}-} +\\
&\qquad + \frac{k}{\mu_{\mathrm{w}}}\rho_{\mathrm{w}} g\left(1-\frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}^2}{\rho_{\mathrm{w}}^2}\right),\\
&mq\rho_{\mathrm{w}} V_{\mathrm{n}} = \lambda_-(\operatorname{grad} T)_{\mathrm{n}-} - \lambda_+(\operatorname{grad} T)_{\mathrm{n}+} - \frac{kq\rho_{\mathrm{w}}}{\mu_{\mathrm{w}}}((\operatorname{grad} P)_{\mathrm{n}-} - \rho_{\mathrm{w}} g),\\
&T_+ = T_- = T_*, \qquad P_+ = P_- = P_*, \qquad \ln\frac{P_*}{P_a} = A + \frac{B}{T_*},\\
&A = 12.512, \qquad B = -4611.73, \qquad P_a = 10^5\,\mathrm{Pa}. \qquad (2)
\end{aligned}$$

The second term in the left-hand side of the heat transfer Equation in (1) describes the heat advective transfer while the right-hand side describes the heat conductive transfer. We consider the flows, where advective transfer can be neglected in comparison with conductive transfer. For the water domain, combining the Darcy's law with the heat conservation equation, we obtain the dimensionless parameter that specifies the ratio of advective and conductive terms:

$$\frac{\rho_{\mathrm{w}} C_{\mathrm{w}}}{\mu_{\mathrm{w}}\lambda_1} k(\delta P - \rho_{\mathrm{w}} g l), \qquad (3)$$

where permeability and pressure may vary strongly. At the same time the other physical parameters vary slightly. Therefore, after substitution of the characteristic values of parameters in (3) the condition of smallness of advective transfer in the water domain can be written in the following form

$$\frac{\rho_{\mathrm{w}} C_{\mathrm{w}}}{\mu_{\mathrm{w}}\lambda_1} k|\delta P - \rho_{\mathrm{w}} g l| \sim 10^{10}\,\mathrm{N}^{-1}\, k|\delta P - \rho_{\mathrm{w}} g l| \ll 1,$$

or

$$k|\delta P - \rho_{\mathrm{w}} g l| \ll 10^{-10}\,\mathrm{N}, \qquad (4)$$

where l is the characteristic length scale.

Analogously, one has for the vapor domain

$$\frac{\rho_v C_p}{\mu_v \lambda_2} k|\delta P - \rho_v g l| \sim 0.33 \times 10^8\,\mathrm{N}^{-1}\, k|\delta P - \rho_v g l| \ll 1,$$

or

$$k|\delta P - \rho_v g l| \ll 3 \times 10^{-9}\,\mathrm{N}. \tag{5}$$

It follows then, that the condition of smallness of the advective transfer in the vapor domain occurs to be weaker than that one in the water domain. We take the difference of values of pressure between high permeability layers as a characteristic variation of pressure, and the distance between these layers as a characteristic length scale. Assuming that (4) and (5) are valid, we ignore the advective heat transfer in the energy equation and obtain usual linear equation of heat transfer for the both domains

$$(\rho C)_{1,2} \frac{\partial T}{\partial t} = \mathrm{div}(\lambda_{1,2}\, \mathrm{grad}\, T). \tag{6}$$

This simplification makes it possible to apply normal-mode analysis of the interface stability in a simple analytic form, while the presence of the advective heat transfer term in (6) requires the use of numerical methods to obtain the linear dispersion equation for perturbations.

If we take the pressure variation in single-phase domains as the characteristic one, then the phase transition regimes can be realized also when the vapor pressure gradient is significantly greater than the water pressure gradient, that is, advective transfer in the vapor domain exceeds advective transfer in the water domain. Consequently, the conditions of smallness of advective transfer have to be considered in each domain separately. In this case a linear size of the corresponding single-phase domain is taken as the characteristic length scale.

3. Stationary Solution

In this section we consider one dimensional problem about stationary phase cross-flow between high permeability layers. As concerns the real situation, the assumption of one-dimensionality is valid if characteristic horizontal length scales of pressure variation in high permeability layers are much greater than the thickness L of the low permeability stratum. Let the upper water-saturated layer $x < 0$ have the pressure P_0 and the temperature T_0, and the lower vapor saturated layer $x > L$ have the pressure P^0 and the temperature T^0. Let us note, that the thickness of the both high permeability layers is not essential for the problem under formulation. It is natural to assume that pressure and temperature in the both high permeability layers are constant (constancy of temperature is guaranteed by heat flow from the surrounding rocks, and constancy of pressure – by the high permeability of the upper and lower layers). Then the phase transition surface, located in the low permeability stratum $0 < x < L$, occupies a certain equilibrium position $x = h$ and the problem has a stationary solution. The domain $0 < x < h$ is saturated by water, and the domain $h < x < L$ – by vapor (Figure 1). Location of the interface $x = h$, and also the values of the temperature T_* and the pressure P_* on this interface are unknown and they have to be found in the process of solving the stationary problem.

Under the condition of smallness of the advective energy transfer the system of equations for the unknown pressure and temperature in the water and vapor domains are given by

$$P''(x) = 0, \qquad T''(x) = 0.$$

Conservation laws of mass and energy on the interface for the case under consideration read

$$\frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{w}}}P'_+ - P'_- + \rho_{\mathrm{w}}g\left(1 - \frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}^2}{\rho_{\mathrm{w}}^2}\right) = 0,$$
$$\lambda_+ T'_+ - \lambda_- T'_- + \frac{kq\rho_{\mathrm{w}}}{\mu_{\mathrm{w}}}(P'_- - \rho_{\mathrm{w}}g) = 0,$$

where prime denotes differentiation with respect to x. These relations, along with the conditions of thermodynamic equilibrium, constitute the full system of boundary conditions across the interface.

The solutions in the water and vapor domains have the form

$$P = P_0 + \frac{P_* - P_0}{h}x, \qquad T = T_0 + \frac{T_* - T_0}{h}x$$
$$P = \frac{P^0 - P_*}{L - h}x + \frac{LP_* - hP^0}{L - h}, \qquad T = \frac{T^0 - T_*}{L - h}x + \frac{LT_* - hT^0}{L - h}, \tag{7}$$

respectively. Substituting (7) into the stationary relations on the interface we get the system of equations for the unknowns T_*, P_* and h:

$$\frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{w}}}\frac{P^0 - P_*}{L - h} - \frac{P_* - P_0}{h} + \rho_{\mathrm{w}}g\left[1 - \frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}^2}{\rho_{\mathrm{w}}^2}\right] = 0,$$
$$\lambda_+\frac{T^0 - T_*}{L - h} - \lambda_-\frac{T_* - T_0}{h} + \frac{kq\rho_{\mathrm{w}}}{\mu_{\mathrm{w}}}\left[\frac{P_* - P_0}{h} - \rho_{\mathrm{w}}g\right] = 0,$$
$$P_* = f(T_*) = P_a \exp\left(A + \frac{B}{T_*}\right). \tag{8}$$

From the first equation the vaporization pressure on the interface can be expressed as a function of dimensionless location parameter $H = h/L$:

$$\frac{P_*}{P_0} = \frac{(P^0/P_0)(c_1/(1 - H)) + 1/H + c_2}{c_1/(1 - H) + 1/H} \equiv F_1(H),$$
$$c_1 = \frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{w}}}, \quad c_2 = \frac{\rho_{\mathrm{w}}gL}{P_0}\left[1 - \frac{\mu_{\mathrm{w}}}{\mu_{\mathrm{v}}}\frac{\rho_{\mathrm{v}}^2}{\rho_{\mathrm{w}}^2}\right].$$

The temperature is expressed from the third equation as a function of the pressure:

$$\frac{T_*}{T_0} = \frac{B/T_0}{\ln P_*/P_0 - \ln P_a/P_0 - A} = \frac{B/T_0}{\ln F_1(H) - \ln P_a/P_0 - A} \equiv F_2(H).$$

This makes it possible to reduce the system to the unique equation for the unknown parameter $H = h/L$

$$\begin{aligned} G(H) \equiv & \left[\frac{T^0}{T_0} - F_2(H)\right]\frac{1}{1-H} - [F_2(H) - 1]\frac{1}{H} + \\ & + \frac{kq\rho_w P_0}{T_0\lambda}\left[\frac{F_1(H) - 1}{H} - \frac{\rho_w g L}{P_0}\right] = 0. \end{aligned} \tag{9}$$

Let us consider the characteristic regimes of stationary flows, or basic states, which are realized in geothermal systems at the following values of parameters: $\mu_w = 1.48 \times 10^{-4}\,\mathrm{Pa\,s}$, $\mu_v = 1.59 \times 10^{-5}\,\mathrm{Pa\,s}$, $\rho_w = 888.66\,\mathrm{kg\,m^{-3}}$, $\rho_v = 4.82\,\mathrm{kg\,m^{-3}}$, $q = 2 \times 10^6\,\mathrm{J/kg}$, $L = 10, 20, 40\,\mathrm{m}$, $C_s = 10^3\,\mathrm{J\,K^{-1}\,kg^{-1}}$, $\lambda_s = 2\,\mathrm{W\,m^{-1}\,K^{-1}}$, $g = 9.8\,\mathrm{m\,s^{-2}}$.

Since for the moderate values of the porosity (for typical geothermal systems the porosity is not large) the heat capacity and the thermal conductivity of the rocks are determined by the corresponding parameters of the porous media skeleton, we put $\lambda_1 = \lambda_2 = \lambda_s$ and $C_1 = C_2 = C_s$ for the sake of simplicity.

Different regimes of phase transitions depending on the values of the temperature and pressure in the high permeability layers can be realized, corresponding to different directions of phase motion. The flow downwards is accompanied by water vaporization and upwards – by vapor condensation.

At $T_0 = 448\,\mathrm{K}$, $T^0 = 464\,\mathrm{K}$, $P_0 = P^0 = 10^6\,\mathrm{Pa}$, $L = 20\,\mathrm{m}$ and $k = 10^{-16}\,\mathrm{m^2}$, for example, we get the following values of the interface parameters $T_* = 456.18\,\mathrm{K}$, $P_* = 1.105 \times 10^6\,\mathrm{Pa}$ and $h = 13.27\,\mathrm{m}$. The water moves downwards under the action of gravity, then finds itself in a domain with greater temperature and vaporizes on the phase transition interface. This leads to the pressure growth on the interface. The pressure there becomes greater than the pressure on the boundaries of the low permeability stratum, but smaller than the hydrostatic one. The arising vapor moves downwards from the phase transition surface and escapes from the vapor saturated domain of the low permeability stratum penetrating into the high permeability layer.

If we increase the pressure in the high permeability layer saturated with vapor up to value $P^0 = 1.1 \times 10^6\,\mathrm{Pa}$, then the solving of the system of equations on the interface gives: $T_* = 454.72$, $P_* = 1.07 \times 10^6\,\mathrm{Pa}$ and $h = 7.9\,\mathrm{m}$. The value of the pressure on the interface now decreases as compared with the previous case, though the pressure on the lower boundary becomes larger. This is due to vapor condensation driven by the motion of vapor upward from the lower high permeability layer and its penetration into the lower temperature zone. Water also moves upward, escaping from low permeability rock into the water-saturated high permeability layer.

The third possibility is also possible, relating to the absence of motion and phase transitions. The case of such an equilibrium is described by the degenerate solution which takes place only if a certain correspondence between the pressure and temperature distributions is observed. This possibility may be interesting from the point of view of the stability properties of the system. It is evident that in the last case the pressure distribution in the equilibrium state has to coincide with a hydrostatic one. The corresponding solution can be constructed by fixing the location of phase transition interface *a priori*.

Let the pressure in the upper water saturated layer be $P_0 = 10^6$ Pa, and the phase interface locate in the middle of the low permeability stratum, that is, $h = L/2$. Then, one can uniquely find the pressure on the interface. The difference between this pressure and the pressure at any boundary exactly equals to the hydrostatic pressure. For $L = 40\,\mathrm{m}$, for example, $P_* = 1.17418 \times 10^6$ Pa. Adding to this value the hydrostatic pressure of the vapor in the vapor saturated domain, we obtain the pressure at the lower boundary at $x = L$: $P^0 = 1.17512 \times 10^6$ Pa. These values of pressure correspond to the dynamical rest state. In order to satisfy the thermodynamic conditions it is necessary to choose the temperature distribution with boundary values lying, respectively, in the water and vapor domains in the Clapeyron plane. Consequently, the temperature at the upper boundary at $x = 0$ must be smaller than the temperature of water boiling: $T_0 < T_f(P_0) = 451.7134\,\mathrm{K}$, and at the lower boundary at $x = L$ temperature must be larger than the vaporization temperature: $T^0 > T_f(P^0) = 458.97\,\mathrm{K}$.

The value of the vaporization temperature T_* on the interface has to belong to the Clapeyron curve at $P_* = 1.17418 \times 10^6$ Pa and therefore this temperature can be uniquely determined and it equals $T_* = 458.93\,\mathrm{K}$. Arbitrariness exists only in a choice of the temperature gradient. For example, putting $T_0 = 450\,\mathrm{K}$ and using the absence of phase transitions (continuity of derivatives of temperature across the interface), and also the known value of the temperature at the interface, we get the following value of the temperature at the lower boundary $x = L$: $T^0 = 467.86\,\mathrm{K}$. Let us note, that this solution is realized at all values of the permeability coefficient.

The dynamic equilibrium or rest state solution takes place at the prescribed boundary values of the pressure and when the temperature coincides with the phase transition temperature on the interface. Variation of the pressure or temperature distribution leads to the onset of phase transition and phase motion for the stationary solution. It means that the rest state solutions represent isolated solutions in the sense that a small variation of the boundary values leads to a solution with phase motion. Depending on whether the pressure decreases or increases, the vaporization or condensation regimes are realized. Accordingly, the location of the interface is displaced downward or upward.

The numerical experiments show that for sufficiently high values of the permeability the Equation (9) has three different roots for the fixed boundary conditions. For example, the rest state solution considered above is unique for

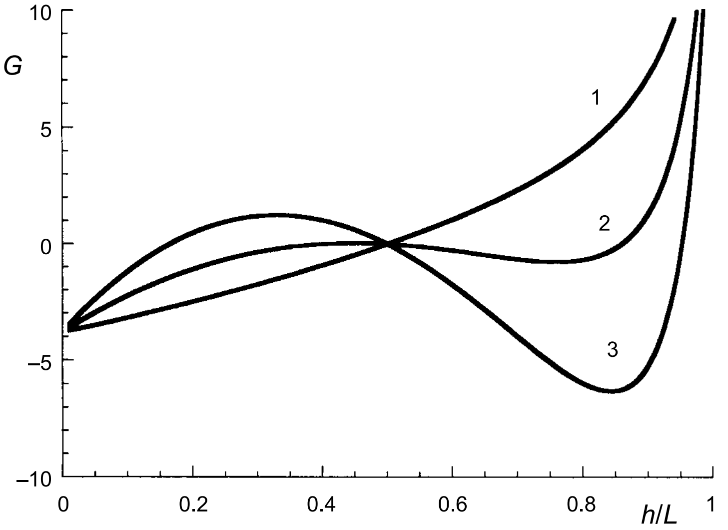

Figure 2. Transcendental function G versus h/L for $L = 40\,\text{m}$, $T_0 = 450\,\text{K}$, $T^0 = 467.86\,\text{K}$, $P_0 = 10^6\,\text{Pa}$, $P^0 = 1.175 \times 10^6\,\text{Pa}$ and three different values of permeability: 1 – $k = 10^{-17}\,\text{m}^2$, 2 – $k = 10^{-16}\,\text{m}^2$, 3 – $k = 2 \times 10^{-16}\,\text{m}^2$.

$k < k_{\text{cr}} \sim 8.9 \times 10^{-17}\,\text{m}^2$. At $k = k_{\text{cr}}$ the bifurcation of the rest state solution takes place, for $k > k_{\text{cr}}$ leading to the originating of two more roots of the Equation (9), corresponding to appearance of the couple of new stationary solutions. In Figure 2 the graph of the transcendental function $G(H)$ is given for three different values of permeability. In the first case (line 1), at $k = 10^{-17}\,\text{m}^2$ the function is monotonous and it intersects the x-axis in a single point $h = 20\,\text{m}$, relating to the rest state. For increasing permeability the curve deforms in a way, that the property of the monotonous behavior ceases to exist and the curve intersects the x-axis in three different points (lines 2,3). It means that apart from the rest state there exist two roots of the Equation (9) at the same boundary values. The smallest root corresponds to a regime with vapor condensation, the central one – to the rest state, and the largest one – to a regime with water vaporization.

The non-uniqueness of the solution makes it necessary the attachment of the additional investigation in order to choose the physically reliable solution. This investigation is usually related to the stability analysis of the solutions in question. Such an analysis is performed in the next section.

The validity of the condition of smallness of advective energy transfer for the prescribed solutions must be also checked. Calculations show that in the first example the conductive heat flux is greater than the advective one by two orders of magnitude in both domains. In the second example the excess is equal to three orders in the water domain and two orders in the vapor domain. For the rest state solution there is no advective energy flux.

4. Stability Analysis

In this section we investigate normal stability of the stationary solution (7) with respect to small perturbations. The system of basic equations, linearized about the equilibrium state solution, under the condition of smallness of advective energy transfer has the form in both domains of the low permeability stratum:

$$\Delta P = 0, \qquad \frac{\partial T}{\partial t} = a_{1,2}\Delta T, \quad a_{1,2} = \frac{\lambda_{1,2}}{(\rho C)_{1,2}} \; 0 < x < h, \; h < x < L. \tag{10}$$

Hereafter, we assume for the sake of simplicity, that the heat capacities and heat conductivities in the both domains are determined by the corresponding parameters of the porous medium skeleton $a = a_1 = a_2$.

The boundary relations for perturbations take the form

$$\begin{aligned}
&P = 0, \qquad T = 0 \quad \text{at } x = 0, L,\\
&P_- = P_+ + \frac{P_0}{L}\Gamma_1\eta, \qquad \Gamma_1 = \frac{1}{H} + \frac{P_1 H - P_f}{H(1-H)} \quad \text{at } x = h,\\
&T_- = T_+ + \frac{T_0}{L}\Gamma_2\eta, \qquad \Gamma_2 = \frac{1}{H} + \frac{T_1 H - T_f}{H(1-H)} \quad \text{at } x = h,\\
&P_- = \left(\frac{\partial f(T)}{\partial T}\right)_{T=T_*} T_- + \left[\left(\frac{\partial f(T)}{\partial T}\right)_{T=T_*}\left(\frac{\partial T}{\partial x}\right)_- - \left(\frac{\partial P}{\partial x}\right)_-\right]\eta\\
&\quad = -\frac{P_0 B}{T_0^2}\Gamma T_- - \frac{P_0}{L}\Gamma_0\eta, \qquad \Gamma = \frac{P_f}{T_f^2},\\
&\Gamma_0 = \frac{B}{T_0}\Gamma\frac{T_f - 1}{H} + \frac{P_f - 1}{H} \quad \text{at } x = h,\\
&m(1-R)\frac{\partial \eta}{\partial t} = \frac{k}{\mu_{\mathrm{v}}}R\left(\frac{\partial P}{\partial x}\right)_+ - \frac{k}{\mu_{\mathrm{w}}}\left(\frac{\partial P}{\partial x}\right)_-, \qquad R = \frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{w}}} \quad \text{at } x = h,\\
&mq\rho_{\mathrm{w}}\frac{\partial \eta}{\partial t} = \lambda_-\left(\frac{\partial T}{\partial x}\right)_- - \lambda_+\left(\frac{\partial T}{\partial x}\right)_+ - \frac{kq\rho_{\mathrm{w}}}{\mu_{\mathrm{w}}}\left(\frac{\partial P}{\partial x}\right)_- \quad \text{at } x = h,
\end{aligned} \tag{11}$$

where $x = h + \eta(t, y)$ is the equation of the interface, $T_f = T_*/T_0$, $T_1 = T^0/T_0$, $P_f = P_*/P_0$, $P_1 = P^0/P_0$.

We look for the expressions of unknown pressure and temperature in the form

$$\{P(x,y,t), T(x,y,t), \eta(y,t)\} = \{\hat{P}(x), \hat{T}(x), \hat{\eta}\}\exp(\hat{\sigma}t + \mathrm{i}\hat{\kappa}y), \tag{12}$$

where $\hat{P}(x)$, $\hat{T}(x)$ are functions on the vertical coordinate x only and $\hat{\eta}$ is a constant. Substituting (12) into (10) we get two pairs of eigenfunctions $\hat{P}_{1,2}(x)$ and $\hat{T}_{1,2}(x)$ in the both domains explicitly, each function depending on two arbitrary constants. These constants along with the constant $\hat{\eta}$ are to be defined via the nine constraints (11). From the condition of existence of a non-trivial solution of (11), we obtain the dispersion relation:

$$\begin{aligned}F(\sigma,\kappa) \equiv\ & \alpha\coth[\alpha(1-H)]\Big[\frac{\omega_{\mathrm{w}}}{a}\Gamma_3\kappa\coth(\kappa H)-(1-R)\sigma+ \\ & +\frac{\omega_{\mathrm{v}}}{a}\Gamma_4R\kappa\coth[\kappa(1-H)]\Big]+\alpha\coth(\alpha H)\Big[\frac{\omega_{\mathrm{v}}}{a}\Gamma_5R\kappa\times \\ & \times\coth[\kappa(1-H)]+\frac{\omega_{\mathrm{w}}}{a}\Gamma_0\kappa\coth(\kappa H)-(1-R)\sigma\Big]+ \\ & +\omega_{\mathrm{v}}\frac{B}{T_0}\frac{mq\rho_{\mathrm{w}}R}{\lambda T_0}\Gamma\kappa\Big[\sigma\coth[\kappa(1-H)]+\sigma\frac{\mu_{\mathrm{v}}}{\mu_{\mathrm{w}}}\coth(\kappa H)+ \\ & +\Gamma_1\kappa\frac{\omega_{\mathrm{w}}}{a}\coth(\kappa h)\coth[\kappa(L-h)]\Big]=0\end{aligned}$$

Here

$$\Gamma_3=\frac{B}{T_0}\Gamma\frac{T_1-T_f}{1-H}+\frac{P_f-1}{H},\qquad \Gamma_4=\frac{B}{T_0}\Gamma\frac{T_1-T_f}{1-H}+\frac{P_1-P_f}{1-H},$$

$$\Gamma_5=\frac{B}{T_0}\Gamma\frac{T_f-1}{H}+\frac{P_1-P_f}{1-H},\qquad \alpha=\sqrt{\kappa^2+\sigma},\qquad \kappa=\hat{\kappa}/L,$$

$$\sigma=a\hat{\sigma}/L^2,\qquad \omega_{\mathrm{w}}=\frac{P_0k}{m\mu_{\mathrm{w}}},\qquad \omega_{\mathrm{v}}=\frac{P_0k}{m\mu_{\mathrm{v}}}$$

$F(\sigma,\kappa)$ is an even function of κ: $F(\sigma,\kappa)= F(\sigma,-\kappa)$, hence the dispersion curve $\sigma=\sigma(\kappa)$ has to be symmetric about σ-axis (reversibility).

The form of the dispersion relation suggests the asymptotics $\sigma\to\sigma_0|\kappa|$ at both infinities, where σ_0 is a constant given by the expression

$$\begin{aligned}\sigma_0 =\ & \left[\frac{\omega_{\mathrm{v}}}{a}R(\Gamma_4+\Gamma_5)+\frac{\omega_{\mathrm{w}}}{a}(\Gamma_0+\Gamma_3)+\frac{\omega_{\mathrm{w}}\omega_{\mathrm{v}}}{a}\frac{B}{T_0}\frac{mq\rho_{\mathrm{w}}R}{\lambda T_0}\Gamma\Gamma_1\right]\times \\ & \times\left[2(1-R)-\frac{\omega_{\mathrm{v}}}{a}\frac{B}{T_0}\frac{mq\rho_{\mathrm{w}}R}{\lambda T_0}\left(1+\frac{\mu_{\mathrm{v}}}{\mu_{\mathrm{w}}}\right)\Gamma\right]^{-1}.\end{aligned}$$

If for any fixed real κ there exist complex roots σ of the dispersion relation $F(\sigma,\kappa)=0$ with positive real part, then the corresponding perturbation will exponentially grow with time, and the basic equilibrium solution (7) will be unstable.

The transcendental function $F(\sigma)=F(\sigma,\kappa)$ for fixed real κ (reversibility implies that one may consider the positive κ only) is analytic everywhere in the complex σ-plane except for the negative real semi-axis, where it has the countable set of poles, bounded away from zero. Zeroes of $F(\sigma)$ correspond to roots of dispersion relation for fixed κ.

The absence of zeroes with positive real part, which give exponential growth to small perturbations, implies stability of the stationary solution in question. For investigation of stability we use the well-known argument principle of complex analysis. According to this principle the difference between the number of zeroes and poles of the function $F(\sigma)$ inside some contour C in the complex σ-plane is equal to the number of rotations of radius-vector in the $F(\sigma)$-complex plane in circuit of a contour C_1, being the image of the contour C for the mapping $F(\sigma)$. We choose a contour C in the σ-plane as the union of the segment $c_0 = \{\sigma = \mathrm{i}z, -R \leqslant z \leqslant R\}$ of the imaginary axis and the semi-circle $c = \{\sigma = R\exp\phi, -\pi/2 < \phi < \pi/2\}$, having the radius R and centered at the origin. The function $F(\sigma)$ has no singularities in the right hand half-plane. If $F(\sigma)$ has zeroes at this half-plane, then by a suitable choice of the large enough radius, it is possible to put all zeroes inside the domain bounded by the contour C.

The contour C transforms into the contour $C_1 = F(c_0) \cup F(c)$ under the mapping $\sigma \rightarrow F(\sigma)$. The components of the contour C_1 are given by the following equations in the complex $F(\sigma)$-plane:

$$F(c_0) = r_1(z)\ \exp[\phi_1(z)], \qquad F(c) = r_2(\phi)\ \exp[\phi_2(\phi)]$$

with certain functions r_i, ϕ_i, $i = 1, 2$.

The form of contours $F(\sigma)$ presented in Figure 3 is typical for absolutely all the regimes under consideration in the present analysis. These contours consist of two parts: the arc *BAC*, being the image $F(c_0)$ of the straight line component of the contour C in the complex σ-plane and the arc *CDB*, being the image $F(c)$ of the semi-circle component of the contour C. In this setting, the principal factor, causing the absence of zeroes of $F(\sigma)$ appears to be the negative value of the non-zero coordinate of the point A. The calculations show that the location of the point A depends on the decay (growth) rate. The smaller the distance from A to origin, the smaller the decay (or growth) rate of perturbations. The first equilibrium state is a stable one (inner contour). When we increase the value of permeability, the configuration becomes unstable. It can be seen that in the first case the roots with positive real part are absent, that implies the stability of the solution. In the second case the image of radius-vector of the points of the contour C makes one round about the origin in the $F(\sigma)$-plane, that means the existence of the root σ with the

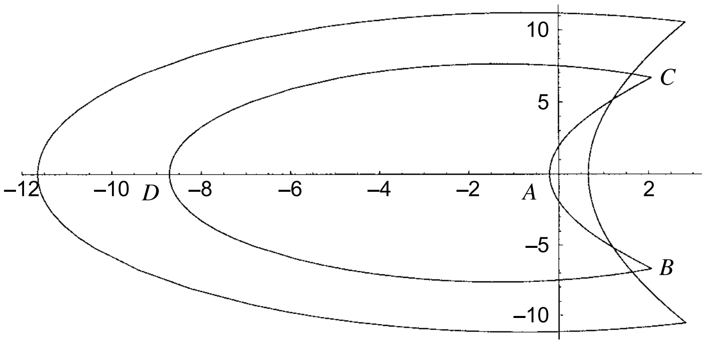

Figure 3. Comparative location of the contours C_1 in the complex $F(\sigma)$-plane for $\kappa = 3$; $k = 10^{-17}\,\mathrm{m}^2$ corresponds to the inner contour and $k = 10^{-16}\,\mathrm{m}^2$ – to the outer contour. The rest of the data are as in Figure 2. The form of the contours varies slightly under variation of κ.

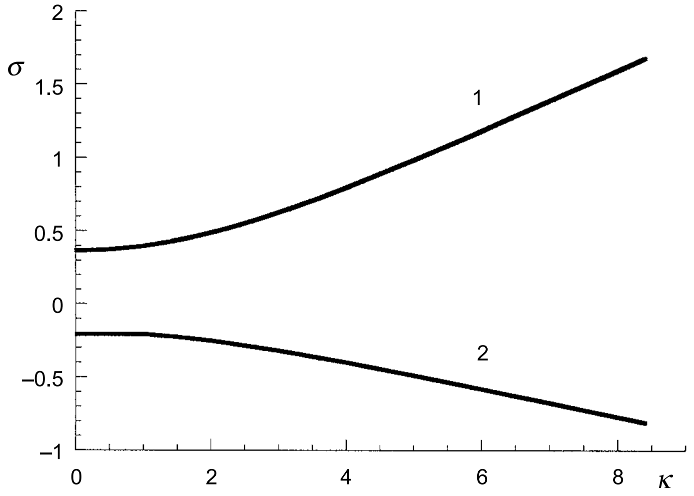

Figure 4. Growth (curve 1) and decay (curve 2) rates σ versus wavenumber κ for the solutions without phase motion. Curves 1 and 2 corresponds to the outer and inner contour in Figure 3.

positive real part and corresponds to the instability regime of the basic solution. The dependence of the decay and growth rates (the corresponding roots are real) on the wavenumber κ is illustrated in Figure 4.

There exists the critical value of permeability $k_{\text{cr}} \sim 8.9 \times 10^{-17}$ m^2 for this solution separating stable and unstable equilibrium states. For the values of permeability lower that the critical one, the solution is stable and it becomes unstable, when the permeability exceeds the threshold k_{cr}.

As it was noted in the preceding section there exist three different roots at $k = 10^{-16}$ m^2 relating to the rest state solution, which was already considered, the solution with water vaporization interface and also the solution with condensation interface, correspondingly. In Figure 5 the dependence of decay (growth) rates relating to the second and the third regimes on κ is shown. As follows from Figure 5, these solutions are stable at small and unstable at large enough wave numbers. We examined the stability of the steady basic state (7) for a lot of the boundary values of the pressure and permeability meeting the criterion of smallness of the advective energy transfer as compared with the conductive one. It was found that for a stable equilibrium state the solution (7) is unique in all cases.

Numerical experiments show, that besides the permeability, other physical parameters can influence the stability. For example, the unstable regimes, demonstrated in Figures 4 and 5, are subjected to qualitative changes when the pressure in the lowest high permeability layers increases from the value $P^0 = 1.17512 \times 10^6$ Pa to $P^0 = 1.2 \times 10^6$ Pa. Initially, there existed three different basic solutions of the unperturbed equations. The solutions, relating to the vapor condensation and water vaporization were stable at small κ and unstable at large enough κ (Figure 5). The solution, relating to the rest state, was unstable for all wave numbers of perturbation.

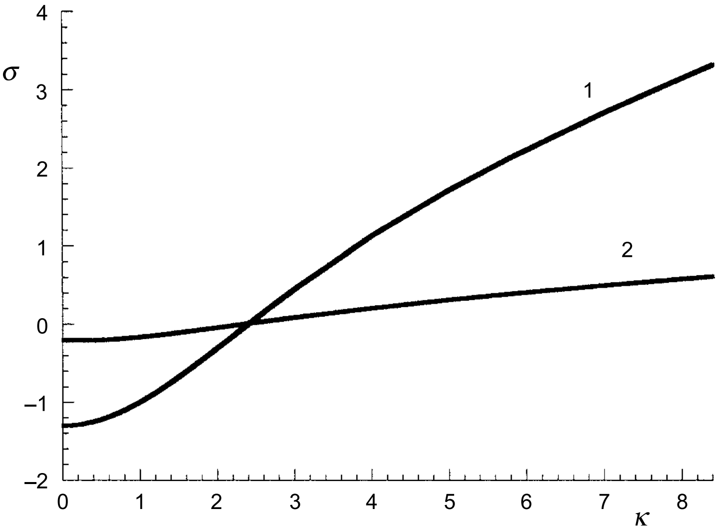

Figure 5. Growth rates σ versus wavenumber κ for the solutions with phase motion. Curve 1 corresponds to the regime of vaporization and curve 2 corresponds to the regime of condensation; $k = 10^{-16}$ m^2. The rest of the data are as in Figures 3 and 4.

Increase of the pressure at the low boundary leads to the uniqueness of the unperturbed basic solution. This solution corresponds to the condensation regime and relates to the unique root of the transcendental equation (9). The solution is stable.

It is interesting to note, that if the pressure in the low layer decreases (for example, $P^0 = 0.99 \times 10^6$ Pa), then also only one root remains, but now it corresponds to the stable regime of vaporization. Consequently, it is reasonable to make a conclusion, that phase motion in the basic stationary solution can stabilize the system. May be the absence of such a flow in the basic solution explains the fact that in the paper (Schubert and Straus, 1980) lower values of critical permeability were found, when the stable configuration is realized, because the rest state occurs to be the most unstable one. This conclusion is also in agreement with data of Eastwood and Spanos (1994), where the comparison of the decay at different wave numbers was performed for an unbounded and bounded geothermal systems and it was found that the presence of the boundary makes the system more stable.

The numerical experiments were also performed in order to find the solutions stable for all wave numbers at higher values of permeability. For the parameters of the system, given in example 1 (see Section 3), growth of the permeability leads to the appearance of three solutions of the unperturbed system, though the condensation regime is stable for an arbitrary wave number. In Figure 6 the dependence of the decay rate on the wave number is shown for $k = 4 \times 10^{-16}$ m^2.

Verification of the smallness of the advective energy transfer in comparison with the conductive one shows that the conductive energy transfer more than by an order of magnitude exceeds the advective heat transfer. The analogous dependence of the

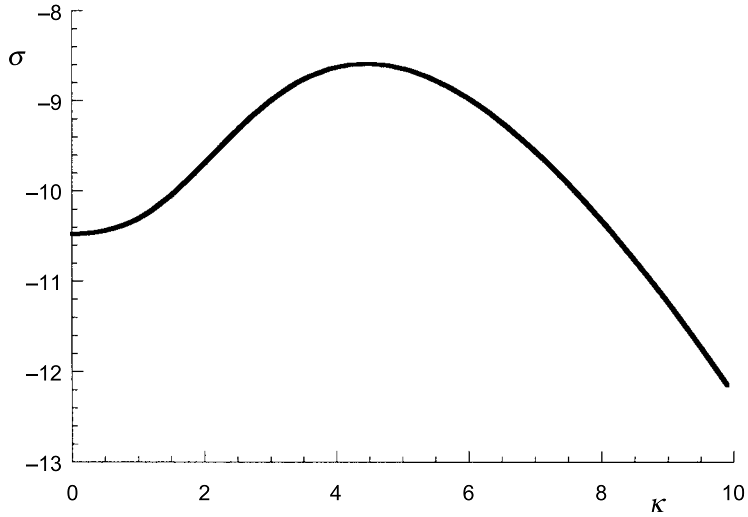

Figure 6. Decay rate σ versus wavenumber κ for the condensation regime at $L = 20\,\mathrm{m}$, $T_0 = 448\,\mathrm{K}$, $T^0 = 464\,\mathrm{K}$, $P_0 = 10^6\,\mathrm{Pa}$, $P^0 = 1.1 \times 10^6\,\mathrm{Pa}$ and $k = 4 \times 10^{-16}\,\mathrm{m}^2$.

decay rate on the wave number, testifying the stability of the solution, takes place also for $k = 6 \times 10^{-16}\,\mathrm{m}^2$, though in this case the conductive energy transfer only three times greater than the advective one. Therefore, increasing the permeability, we get either the transfer to instability, or find ourselves without the frame of our approximation.

Let us note, that in Schubert and Straus (1980) and Eastwood and Spanos (1994) it is assumed that the roots are purely real, not complex. This assumption, generally speaking, requires the substantiation. In our analysis we are able to detect also the complex roots σ of the dispersion relation with positive real part. The existence of such roots also implies the instability of the physical system under investigation.

5. Conclusion and Discussion

In the present paper we find the stationary solution of the problem of phase motion in a geothermal system where the water layer overlies the vapor layer under the assumption of the smallness of the advective energy transfer as compared with the conductive one. It is found that for low values of the permeability the basic solution is unique. If the value of permeability exceeds some critical value, there exist three different locations of the phase transition surface. It means that for the same boundary values three different regimes of phase motion can be realized.

The analysis of normal stability of basic solutions shows, that in the cases when the solution is unique, it is always stable. If the basic solution is not unique, there exist three different solutions and the question about the stability of each solution has to be treated separately. In this case the regime with the intermediate location of the interface occurs to be the most unstable one, and the other two regimes can be either stable or unstable. In some cases the condition of smallness of the advective energy transfer is violated before the bifurcation of basic state takes place and therefore, the criterion of stability coincides with the condition of smallness of the advective heat transfer. This fact makes it possible to treat the physical mechanism of the stability in the sense that perturbations of the interface, formation and penetration of water fingers in the vapor domain are suppressed by the dominating conductive heat flux, which leads to the vaporization of the liquid phase. The calculations show that there exist stable basic states for the values of permeability $k \sim 0.6 \times 10^{-15}\,\mathrm{m}^2$ and higher. These values exceed more than by an order of magnitude the critical value, given in Schubert and Straus (1980). At higher values of the permeability the role of the advective heat transfer becomes considerable and the use of the solution of the unperturbed system is not rightful. In cases when the violation of our criterium precedes the transfer to instability, it seems to be reasonable to conjecture the growth of permeability will not lead to the transition of the system to instability immediately, i.e. in reality the water layer can exist over the vapor layer even for higher values of the rock permeability.

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