http://www.mapleprimes.com/questions/149180-BUG-In-Numerical-Integration-With-SphericalYs

```
> restart; interface(version);
  Digits:=15;
```

Classic Worksheet Interface, Maple 17.00, Windows, Feb 21 2013, Build ID 813473

$$Digits := 15$$

```
> [theta=0..Pi,phi=0..2*Pi];
```

$$[\theta=0\,..\,\pi, \phi=0\,..\,2\,\pi]$$

```
> #Digits:=3:
> chi:=Pi/2-theta;
  Vsi:='-(1/sqrt( 1+epsilon - cos(chi)*cos(phi-Pi/3)) + 1/sqrt( 1 +epsilon- cos(chi)*cos(phi-Pi))+
  1/sqrt(1+epsilon-cos(chi)*cos(phi+Pi/3)))';
  Vsi:=%;
```

$$\chi := \frac{\pi}{2}-\theta$$

$$Vsi := -\frac{1}{\sqrt{1+\varepsilon-\cos(\chi)\cos\left(\phi-\frac{\pi}{3}\right)}}-\frac{1}{\sqrt{1+\varepsilon-\cos(\chi)\cos(\phi-\pi)}}-\frac{1}{\sqrt{1+\varepsilon-\cos(\chi)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

$$Vsi := -\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

```
>
```

Vsi

```
> eval(Vsi, epsilon = 10^(-7));
  % = evalc(Re(%)) assuming 0<theta,theta<Pi,0<phi,phi<2*Pi; is(%);
```

$$-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

$$-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}=$$
$$-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

$$true$$

```
> eval(Vsi, epsilon = 10^(-7));
  evalc(Im(%)) assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
```

$$-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

$$0$$

```
> eval(Vsi, epsilon = 10^(-7));
  [op(%)]; map(Im, %)  assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
```

$$-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

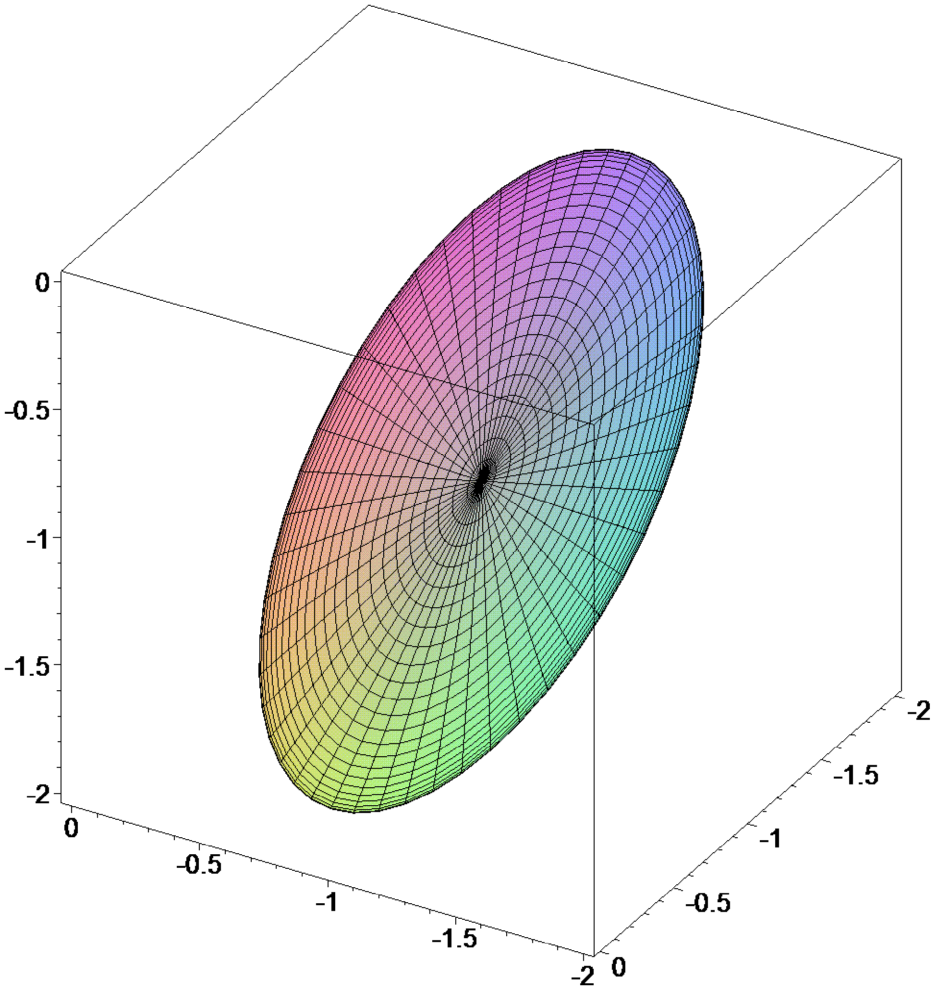
$$\left[-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}, -\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}, -\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right]$$

$$[0, 0, 0]$$

```
> Q:=[1+epsilon-sin(theta)*sin(phi+1/6*Pi),1+epsilon+sin(theta)*cos(phi),1+epsilon-sin(theta)*cos(phi
  +1/3*Pi)];
  EPS:=map(solve,%, epsilon);
  plot3d(%, theta=0..Pi,phi=0..2*Pi, axes=boxed, orientation=[120,60]);
```

$$Q := \left[1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right), 1+\varepsilon+\sin(\theta)\cos(\phi), 1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)\right]$$

$$EPS := \left[\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)-1, -\sin(\theta)\cos(\phi)-1, \sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)-1\right]$$




```
> map('q -> is(q <=0)', EPS)   assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
```

$$[\mathrm{true}, \mathrm{true}, \mathrm{true}]$$

```
> map('q -> is(0<=q)', Q)   assuming 0<theta,theta<Pi,0<phi,phi<2*Pi, 0 <= epsilon;
```

$$[\mathrm{true}, \mathrm{true}, \mathrm{true}]$$

So we simply need 0 < epsilon

```
>
>
```

simplify integrand

```
> SP1:=SphericalY(3,-3,theta,phi)*conjugate(SphericalY(0,0,theta,phi))*sin(theta);
```

$$SP1 := \mathrm{SphericalY}(3, -3, \theta, \phi)\, \overline{(\mathrm{SphericalY}(0, 0, \theta, \phi))}\, \sin(\theta)$$

```
> Vsi*SP1;
  convert(%, hypergeom);
```

$$\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\mathrm{SphericalY}(3, -3, \theta, \phi)\, \overline{(\mathrm{SphericalY}(0, 0, \theta, \phi))}\, \sin(\theta)$$

$$-\frac{1}{24}\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(\cos(\theta)-1)^{(3/2)}$$

$$\mathrm{hypergeom}\left([-3],[\ ],\frac{1}{2}-\frac{1}{2}\cos(\theta)\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\, e^{(3\,I\,\phi)}\,(\cos(\theta)+1)^{(3/2)}\right)$$

```
> hypergeom([-3],[],1/2-1/2*cos(theta));
  eq:= % = simplify(%);
```

$$\mathrm{hypergeom}\left([-3],[\ ],\frac{1}{2}-\frac{1}{2}\cos(\theta)\right)$$

$$eq := \mathrm{hypergeom}\left([-3],[\ ],\frac{1}{2}-\frac{1}{2}\cos(\theta)\right)=\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3$$

```
> SP1;
  convert(%, hypergeom);
  eval(%, eq);
```

$$\mathrm{SphericalY}(3,-3,\theta,\phi)\,\overline{(\mathrm{SphericalY}(0,0,\theta,\phi))}\,\sin(\theta)$$

$$-\frac{1}{24}\,\frac{(\cos(\theta)-1)^{(3/2)}\,\mathrm{hypergeom}\left([-3],[\ ],\frac{1}{2}-\frac{1}{2}\cos(\theta)\right)\sqrt{5040}\,\sin(\theta)}{\pi\, e^{(3\,I\,\phi)}\,(\cos(\theta)+1)^{(3/2)}}$$

$$-\frac{1}{24}\,\frac{(\cos(\theta)-1)^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)}{\pi\, e^{(3\,I\,\phi)}\,(\cos(\theta)+1)^{(3/2)}}$$

```
> Vsi*SP1;
  convert(%, hypergeom);
  eval(%, eq);
```

$$\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\mathrm{SphericalY}(3,-3,\theta,\phi)\,\overline{(\mathrm{SphericalY}(0,0,\theta,\phi))}\,\sin(\theta)$$

$$-\frac{1}{24}\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(\cos(\theta)-1)^{(3/2)}\,\mathrm{hypergeom}\left([-3],[\ ],\frac{1}{2}-\frac{1}{2}\cos(\theta)\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\, e^{(3\,I\,\phi)}\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$-\frac{1}{24}\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(\cos(\theta)-1)^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\, e^{(3\,I\,\phi)}\,(\cos(\theta)+1)^{(3/2)}\right)$$

```
> (cos(theta)-1)^(3/2);
  eq2:=% =evalc(%)  assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
  rhs(%): [Re(%), Im(%)] assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
```

$$(\cos(\theta)-1)^{(3/2)}$$

$$eq2 := (\cos(\theta)-1)^{(3/2)} = -I\,(1-\cos(\theta))^{(3/2)}$$

$$[0,-(1-\cos(\theta))^{(3/2)}]$$

```
> Vsi*SP1:
  convert(%, hypergeom):
  eval(%, eq):
  eval(%, eq2);
  Kc:=combine(%, power);
```

$$\frac{1}{24}I\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\, e^{(3\,I\,\phi)}\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$Kc := \frac{1}{24}I\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)e^{(-3\,I\,\phi)}\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

```
>
>
> K:=eval(Kc, epsilon = 10^(-7));
```

$$K := \frac{1}{24}I\left(-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)e^{(-3\,I\,\phi)}$$

$$(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

```
> 'Kc': '%'=%;
```

$$Kc=\frac{1}{24}I\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)e^{(-3\,I\,\phi)}(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

```
> 'Int(Vsi*(SP1), [theta=0..Pi,phi=0..2*Pi])' = 'Int(Kc, [theta=0..Pi,phi=0..2*Pi])';
```

$$\int_0^{2\pi}\int_0^{\pi} Vsi\ SP1\,d\theta\,d\phi=\int_0^{2\pi}\int_0^{\pi} Kc\,d\theta\,d\phi$$

real and imaginary part

```
> (1-cos(theta))^(3/2)*1/(cos(theta)+1)^(3/2);
  evalc(Im(%))  assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
```

$$\frac{(1-\cos(\theta))^{(3/2)}}{(\cos(\theta)+1)^{(3/2)}}$$

$$0$$

```
> 1/8+3/8*cos(theta)+3/8*cos(theta)^2+1/8*cos(theta)^3;
  evalc(Im(%))  assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
  #plot(%%, theta=0 .. Pi);
```

$$\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3$$

$$0$$

```
> exp(-3*I*phi);
  %=evalc(%);
```

$$e^{(-3\,I\,\phi)}$$

$$e^{(-3\,I\,\phi)}=\cos(3\,\phi)-\sin(3\,\phi)\,I$$

```
> Kr:=subs(exp(-3*I*phi)=-sin(3*phi)*I, Kc);
  Ki:=subs(exp(-3*I*phi)=cos(3*phi),    Kc) / I;
```

$$Kr:=\frac{1}{24}\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\sin(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$Ki:=\frac{1}{24}\left(-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\cos(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

```
>
> # do a simple check for typos/errors before proceeding:

  eval(Kr, epsilon = 1e-7);
  Int(%, [theta=0..Pi,phi=0..2*Pi], method = _cuhre, digits = 7): evalf(%);

  eval(Ki, epsilon = 1e-7);
  Int(%, [theta=0..Pi,phi=0..2*Pi], method = _cuhre, digits = 9): evalf(%);
```

$$\frac{1}{24}\left(-\frac{1}{\sqrt{1.0000001-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1.0000001+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1.0000001-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\sin(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$-0.2010545\ 10^{-14}$$

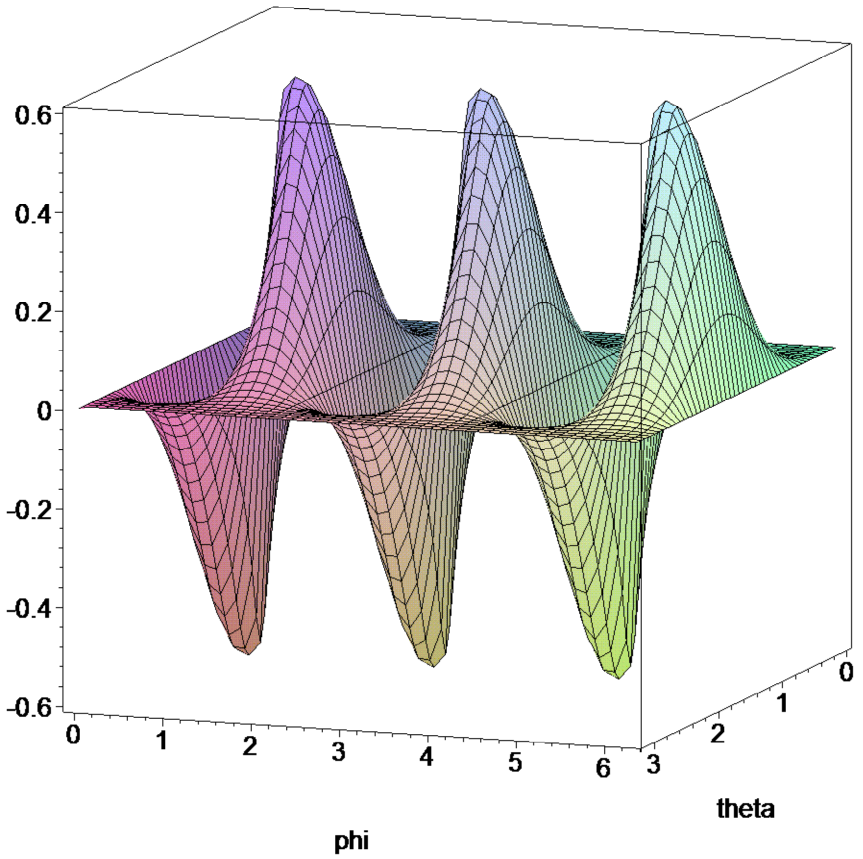
$$\frac{1}{24}\left(-\frac{1}{\sqrt{1.0000001-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1.0000001+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1.0000001-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\cos(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$0.895019433$$

```
>
>
```

reduce the task by symmetries, show that real part evaluates to 0

```
> Re(K): plot3d(%, theta=0..Pi,phi=0..2*Pi, axes=boxed, orientation=[20,80]);
```




```
> RK:=Re(K) assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
  % =  eval(%, phi = phi + 2*Pi/3); is(%);
```

$$RK := -\left(\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}+\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}+\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)$$
$$(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\sin(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$-\left(\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}+\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}+\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}$$
$$\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\sin(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)=-$$
$$\left(\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}+\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}+\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}$$
$$\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\sin(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

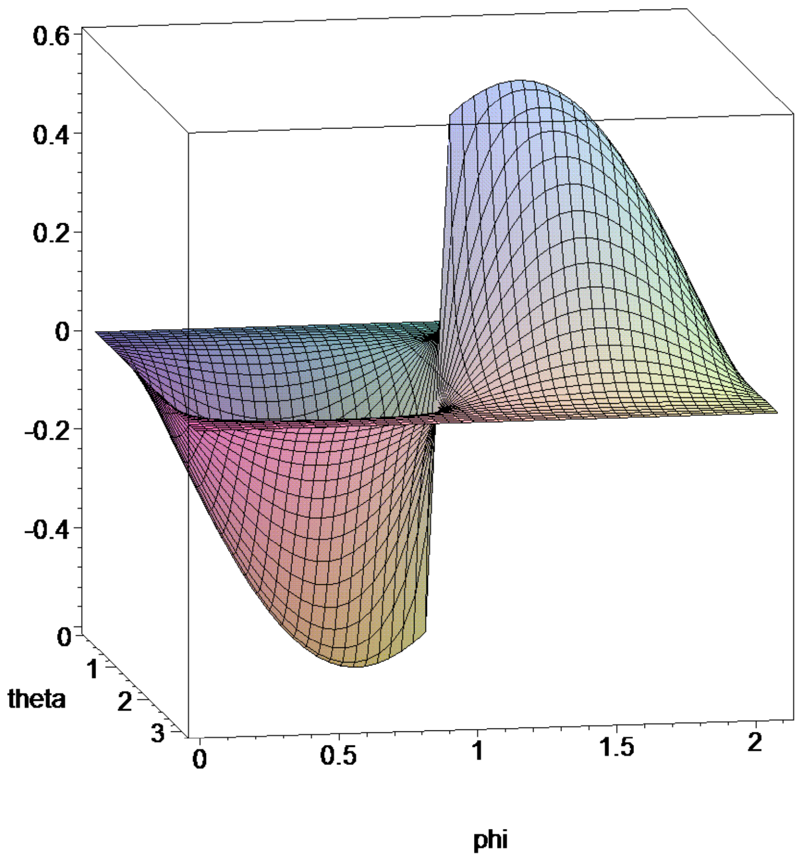
true

```
> 'Kr':
  '% =  eval(%, phi = phi + 2*Pi/3)'; is(%);
```

$$Kr = Kr\Big|_{\phi = \phi + \frac{2\pi}{3}}$$

true

```
> plot3d(RK, theta=0..Pi,phi=0..2*Pi/3, axes=boxed, orientation=[-10,80]);
```




```
> 'eval( RK, phi = 2*Pi/3 / 2 -t)+ eval( RK, phi = 2*Pi/3 / 2 +t)';
  %;
```

$$RK\Big|_{\phi = \frac{\pi}{3} - t} + RK\Big|_{\phi = \frac{\pi}{3} + t}$$

$$0$$

```
> 'eval( Kr, phi = 2*Pi/3 / 2 -t)+ eval( Kr, phi = 2*Pi/3 / 2 +t)';
  %;
```

$$Kr\Big|_{\phi = \frac{\pi}{3} - t} + Kr\Big|_{\phi = \frac{\pi}{3} + t}$$

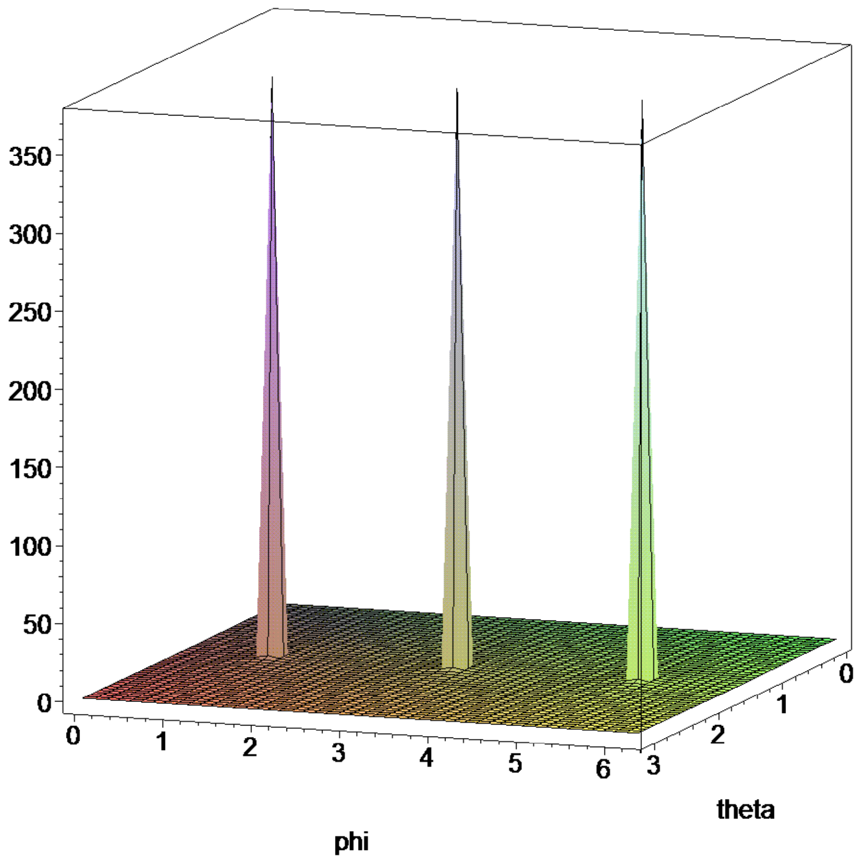
$$0$$

Hence integral RK w.r.t. phi over 0 .. 2*Pi / 3 is zero.

```
>
> IK:=Im(K) assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
  plot3d(%, theta=0..Pi,phi=0..2*Pi, axes=boxed, orientation=[20,80]);
```

$$IK := \left(-\frac{1}{24} \frac{1}{\sqrt{\frac{10000001}{10000000} - \sin(\theta)\sin\left(\phi + \frac{\pi}{6}\right)}} - \frac{1}{24} \frac{1}{\sqrt{\frac{10000001}{10000000} + \sin(\theta)\cos(\phi)}} - \frac{1}{24} \frac{1}{\sqrt{\frac{10000001}{10000000} - \sin(\theta)\cos\left(\phi + \frac{\pi}{3}\right)}} \right)$$

$$(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\cos(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$




```
> IK;
  % =  eval(%, phi = phi + 2*Pi/3); is(%);
```

$$\left(-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}$$
$$\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\cos(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$\left(-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}$$
$$\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\cos(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)=$$
$$\left(-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}$$
$$\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\cos(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

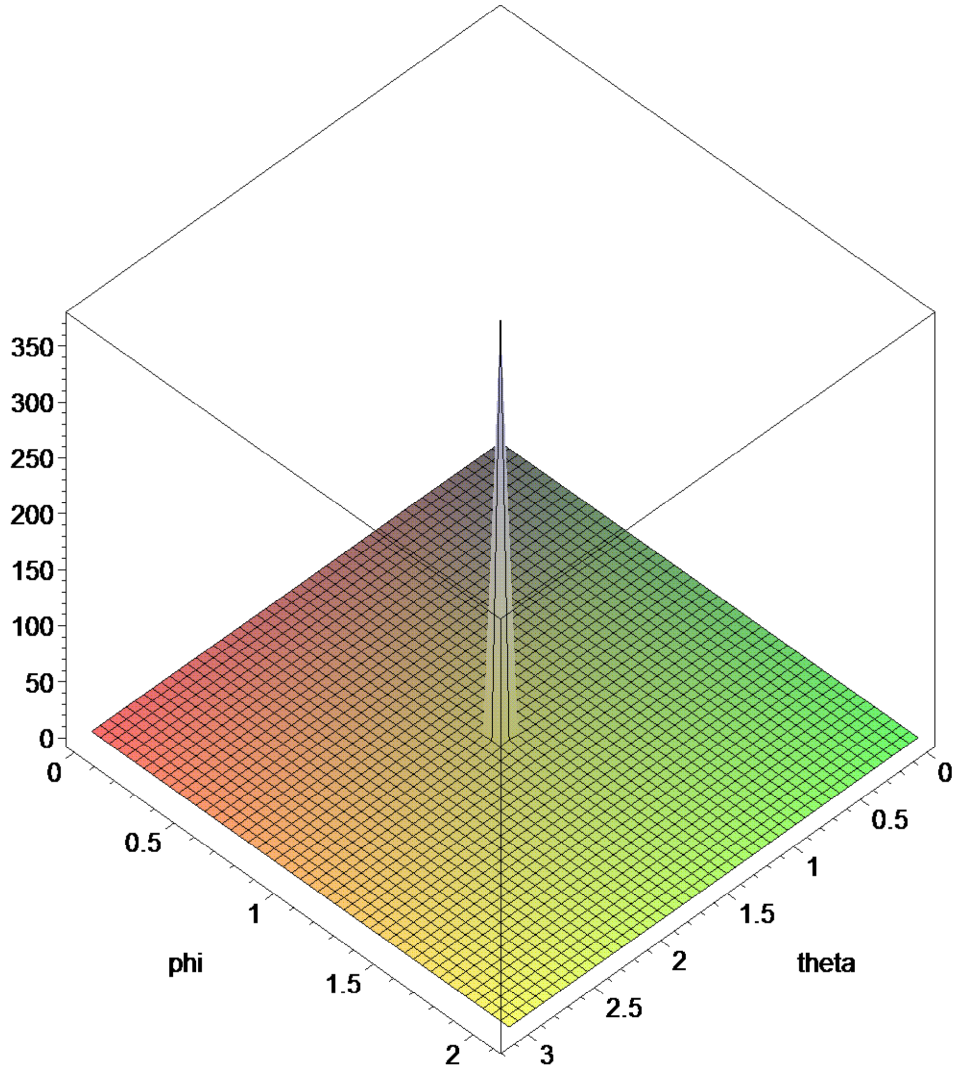
true

```
> 'Ki':
  '% =  eval(%, phi = phi + 2*Pi/3)'; is(%);
```

$$Ki = Ki\Big|_{\phi = \phi + \frac{2\pi}{3}}$$

true

```
> plot3d(IK, theta=0..Pi,phi=0..2*Pi/3, axes=boxed);
```




```
> 'eval( IK, phi = 2*Pi/3 / 2 -t) - eval( IK, phi = 2*Pi/3 / 2 +t)';
  %;
```

$$IK\Big|_{\phi = \frac{\pi}{3} - t} - IK\Big|_{\phi = \frac{\pi}{3} + t}$$

0

```
> 'eval( Ki, phi = 2*Pi/3 / 2 -t) - eval( Ki, phi = 2*Pi/3 / 2 +t)';
  %;
```

$$Ki\Big|_{\phi = \frac{\pi}{3} - t} - Ki\Big|_{\phi = \frac{\pi}{3} + t}$$

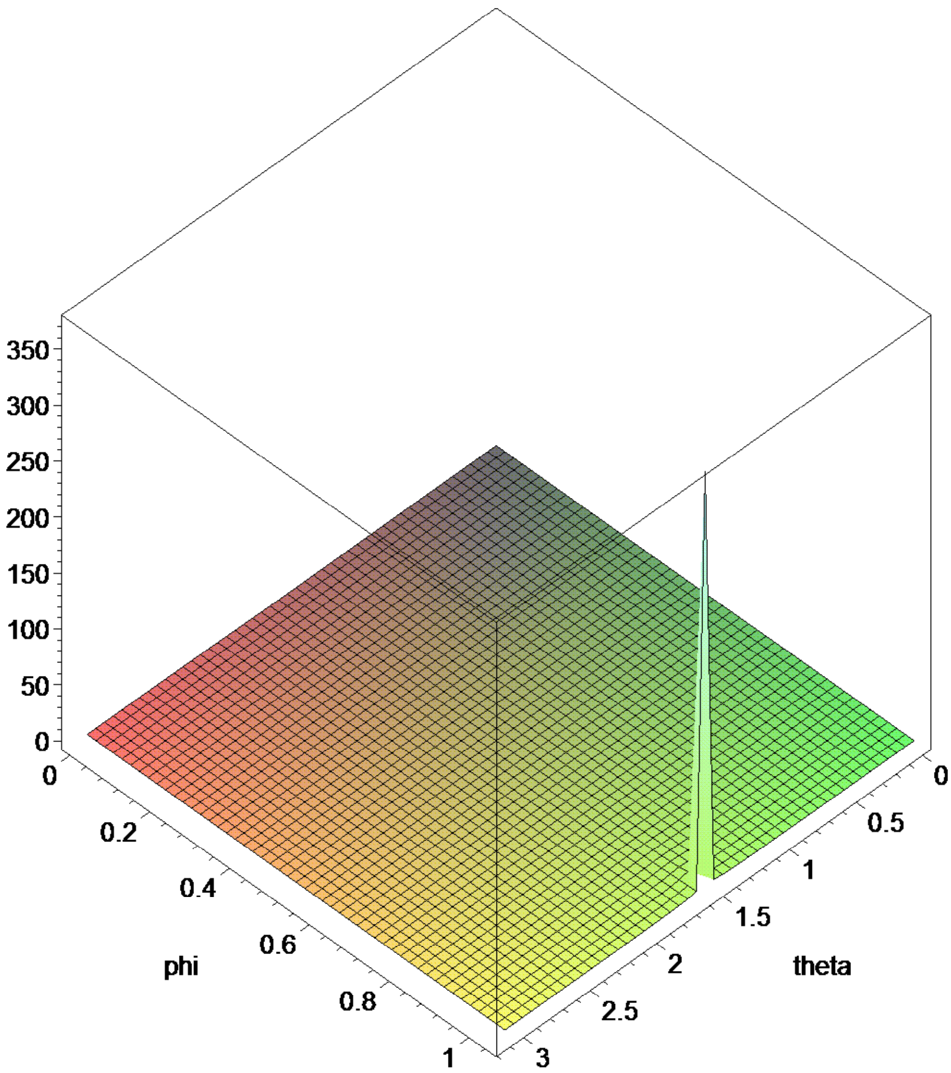
0

```
> IK;
  rationalize(%): #simplify(%); #int(%, phi):
  6*Int(%, [theta=0..Pi,phi=0..2*Pi/3/2], method = _cuhre, digits = 9): evalf(%);
```

$$\left(-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000} - \sin(\theta)\sin\left(\phi + \frac{\pi}{6}\right)}} - \frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000} + \sin(\theta)\cos(\phi)}} - \frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000} - \sin(\theta)\cos\left(\phi + \frac{\pi}{3}\right)}}\right)(1 - \cos(\theta))^{(3/2)}$$
$$\left(\frac{1}{8} + \frac{3}{8}\cos(\theta) + \frac{3}{8}\cos(\theta)^2 + \frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\cos(3\,\phi) \Big/ \left(\pi\,(\cos(\theta) + 1)^{(3/2)}\right)$$

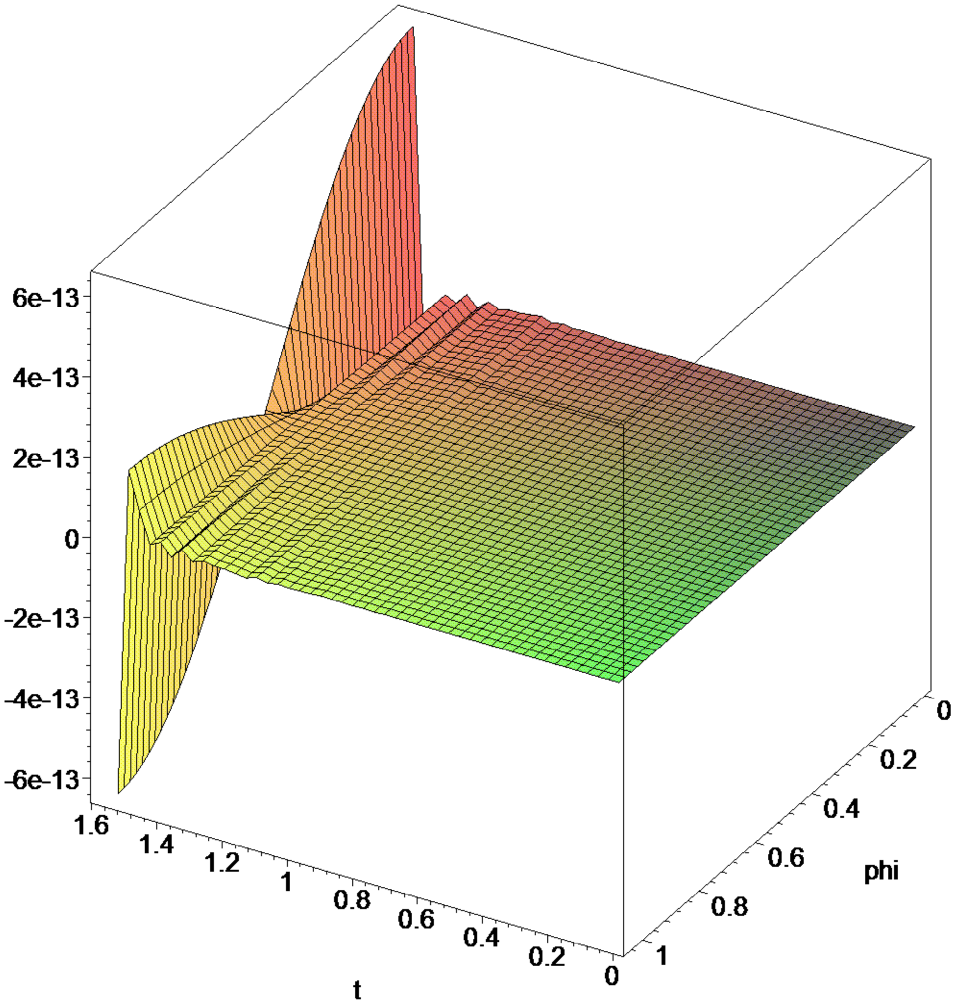
$$0.895019436$$

```
> plot3d(IK, theta=0..Pi,phi=0..2*Pi/3/2,  axes=boxed);
```




```
> 'eval(IK, theta= Pi/2 - t) - eval(IK, theta= Pi/2 + t)';
  #% assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
  plot3d(%, t=0..Pi/2,phi=0..2*Pi/3/2,  axes=boxed, orientation=[120,60]);;
```

$$IK\Big|_{\theta=\frac{\pi}{2}-t} - IK\Big|_{\theta=\frac{\pi}{2}+t}$$




```
> 'eval(Ki, theta= Pi/2 - t) - eval(Ki, theta= Pi/2 + t)';
  simplify(%) assuming 0<theta,theta<Pi,0<phi,phi<2*Pi;
```

$$\left.Ki\right|_{\theta=\frac{\pi}{2}-t}-\left.Ki\right|_{\theta=\frac{\pi}{2}+t}$$

$$0$$

```
> 
> 
```

compute up to 12 decimals

```
> IK;
  rationalize(%): #simplify(%); #int(%, phi):
  2*6*Int(%, [theta=0..Pi/2,phi=0..2*Pi/3/2], method = _cuhre, digits = 9): evalf(%);
```

$$\left(-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{24}\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\cos(3\,\phi)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$0.8950194336$$

```
> 'Ki'; rationalize(%):
  Q:=simplify(%, size) assuming 0<theta,theta<Pi/2,0<phi,phi<2*Pi/3, 0<epsilon;
  #combine(%, power)  assuming 0<theta,theta<Pi/2,0<phi,phi<2*Pi/3, 0<epsilon;
```

$$Ki$$

$$Q := -\frac{1}{192}\cos(3\,\phi)\,\sqrt{5040}\,\sin(\theta)\,(\cos(\theta)+1)^{(3/2)}\Bigg(\Big(\big(\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}+\sqrt{1+\varepsilon-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\big)\sqrt{1+\varepsilon-\sin(\theta)\cos\big(\phi+\tfrac{\pi}{3}\big)} + \sqrt{1+\varepsilon-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\,\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}\Big)(1-\cos(\theta))^{(3/2)} \Big/ \Big(\sqrt{1+\varepsilon-\sin(\theta)\cos\big(\phi+\tfrac{\pi}{3}\big)}\,\sqrt{1+\varepsilon-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\,\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}\,\pi\Big)\Bigg)$$

```
> eval(Q,epsilon=1e-7);
  2*6*Int(%, [theta=0..Pi/2, phi=0..2*Pi/3/2], method = _cuhre, digits = 12);
  evalf(%);
```

$$-\frac{1}{192}\cos(3\,\phi)\,\sqrt{5040}\,\sin(\theta)\,(\cos(\theta)+1)^{(3/2)}\Bigg(\Big(\big(\sqrt{1.0000001+\sin(\theta)\cos(\phi)}+\sqrt{1.0000001-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\big)\sqrt{1.0000001-\sin(\theta)\cos\big(\phi+\tfrac{\pi}{3}\big)} + \sqrt{1.0000001-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\,\sqrt{1.0000001+\sin(\theta)\cos(\phi)}\Big)(1-\cos(\theta))^{(3/2)} \Big/ \Big(\sqrt{1.0000001-\sin(\theta)\cos\big(\phi+\tfrac{\pi}{3}\big)}\,\sqrt{1.0000001-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\,\sqrt{1.0000001+\sin(\theta)\cos(\phi)}\,\pi\Big)\Bigg)$$

$$12\,\mathrm{Int}\Bigg(-\frac{1}{192}\cos(3\,\phi)\,\sqrt{5040}\,\sin(\theta)\,(\cos(\theta)+1)^{(3/2)}\Bigg(\Big(\big(\sqrt{1.0000001+\sin(\theta)\cos(\phi)}+\sqrt{1.0000001-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\big)\sqrt{1.0000001-\sin(\theta)\cos\big(\phi+\tfrac{\pi}{3}\big)} + \sqrt{1.0000001-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\,\sqrt{1.0000001+\sin(\theta)\cos(\phi)}\Big)(1-\cos(\theta))^{(3/2)} \Big/ \Big(\sqrt{1.0000001-\sin(\theta)\cos\big(\phi+\tfrac{\pi}{3}\big)}\,\sqrt{1.0000001-\sin(\theta)\sin\big(\phi+\tfrac{\pi}{6}\big)}\,\sqrt{1.0000001+\sin(\theta)\cos(\phi)}\,\pi\Big)\Bigg), \Big[\theta = 0\,..\,\frac{\pi}{2}, \phi = 0\,..\,\frac{\pi}{3}\Big], \mathrm{method} = \_\mathrm{cuhre}, \mathrm{digits} = 12\Bigg)$$

$$0.8950194330360$$

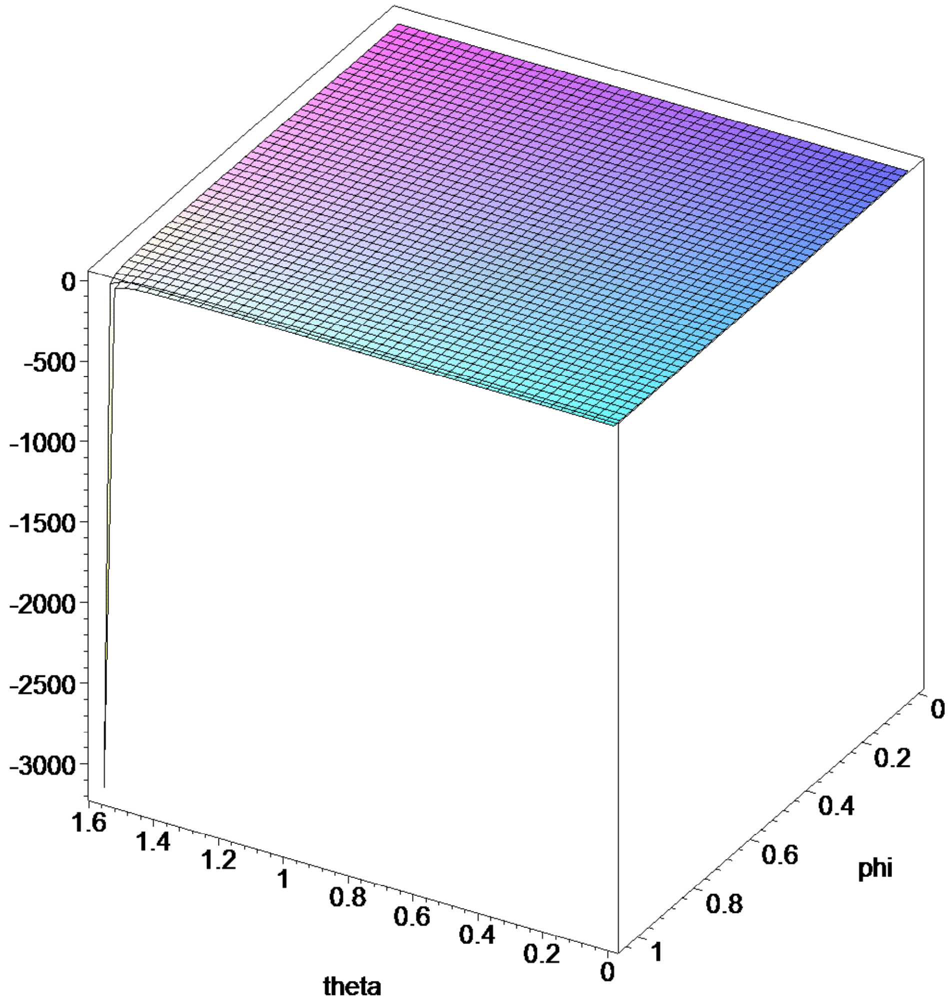
```
> 
> 
```

improve

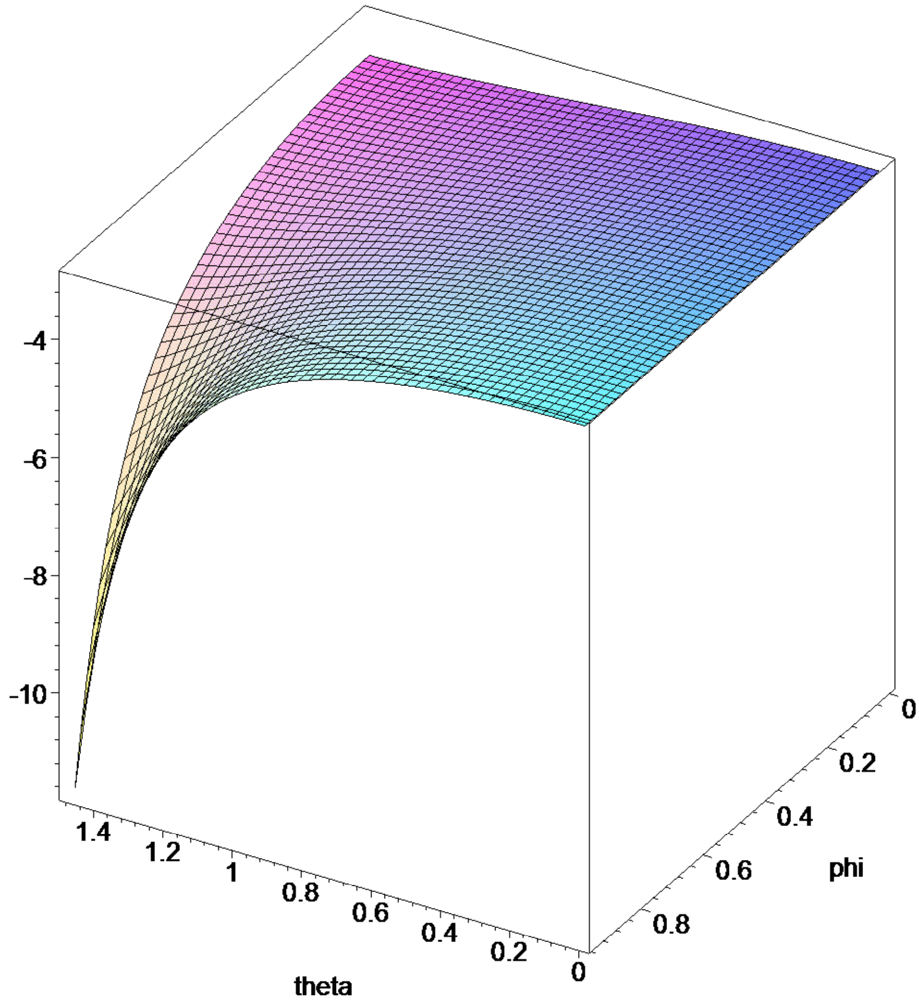
```
> 'Vsi'; %; eval(%, epsilon = 10^(-7)):
  plot3d(%, theta=0..Pi/2,phi=0..2*Pi/3/2,  axes=boxed, orientation=[120,60]);

  Vsi: eval(%, epsilon = 10^(-7)):
  plot3d(%, theta=0..Pi/2 - 1e-1,phi=0..2*Pi/3/2 - 1e-1,  axes=boxed, orientation=[120,60]);
```

$$\mathrm{Vsi}$$

$$-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\big(\phi+\frac{\pi}{6}\big)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\big(\phi+\frac{\pi}{3}\big)}}$$



0
-500
-1000
-1500
-2000
-2500
-3000
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
theta
0
0.2
0.4
0.6
phi
0.8
1




```
> 
> Vsi; [op(%)];
  subs(theta=Pi/2, %): subs(phi=Pi/3, %): simplify(%); eval(%, epsilon = 1e-7);
```

$$-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

$$\left[-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}},-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}},-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right]$$

$$\left[-\frac{1}{\sqrt{\varepsilon}},-\frac{2}{\sqrt{6+4\,\varepsilon}},-\frac{2}{\sqrt{6+4\,\varepsilon}}\right]$$

[-3162.27766016838, -0.816496553711176, -0.816496553711176]

So care for the "numerical singularity"

```
> Vs, V2, V3:= op(Vsi);

  'Vsi = Vs + (V2+V3)'; is(%);
```

$$Vs, V2, V3 := -\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}},-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}},-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$

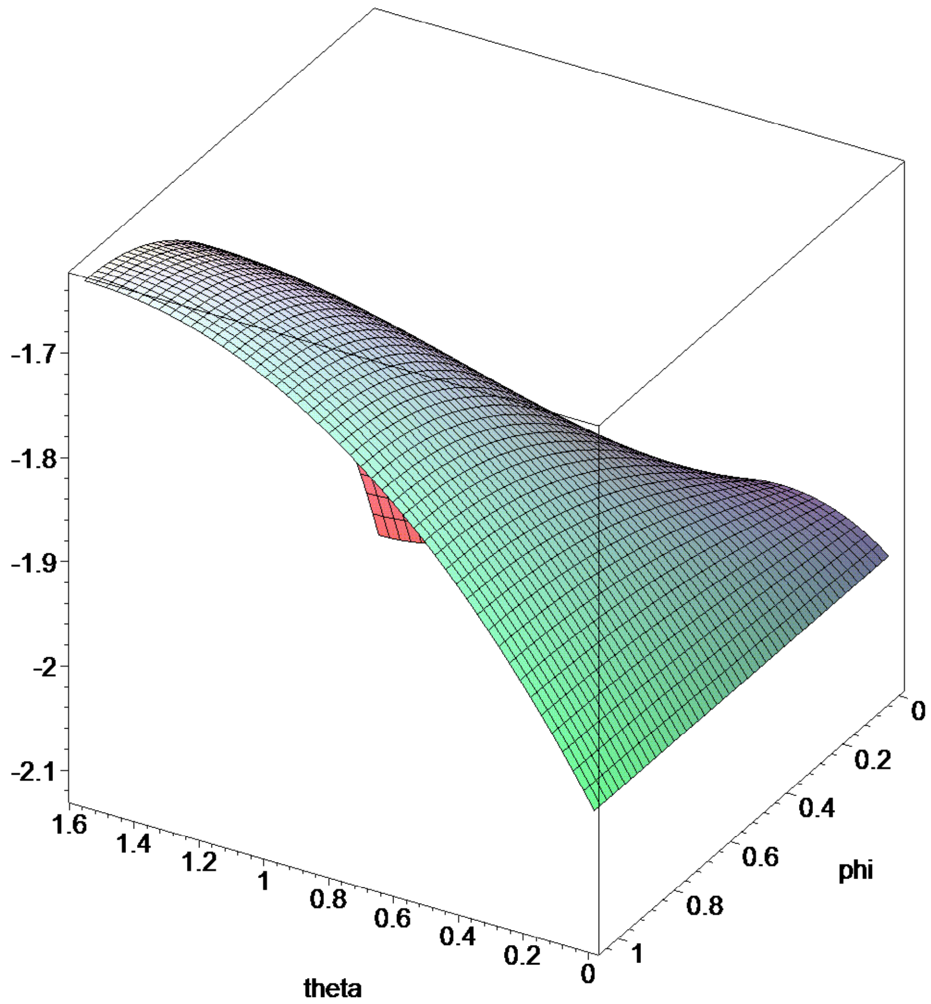
$$Vsi = Vs + V2 + V3$$

true

```
> 'V2+V3'; eval(%, epsilon = 10^(-7));
  plot3d(%, theta=0..Pi/2,phi=0..2*Pi/3/2,  axes=boxed, orientation=[120,60]);
```

$$V2+V3$$

$$-\frac{1}{\sqrt{\frac{10000001}{10000000}+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}$$




```
> # this uses 15 Digits !
  Ki/Vsi* (V2+V3);
  eval(%,epsilon=1e-7);
  2*6*Int(%, [theta=0..Pi/2, phi=0..2*Pi/3/2]); #, method = _cuhre, digits = 12);
  regularPart:=evalf(%);
```

$$\frac{1}{24}\cos(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\left(-\frac{1}{\sqrt{1+\varepsilon+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$\frac{1}{24}\cos(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^2+\frac{1}{8}\cos(\theta)^3\right)\sqrt{5040}\,\sin(\theta)\left(-\frac{1}{\sqrt{1.0000001+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1.0000001-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)$$

$$12\int_{0}^{\frac{\pi}{3}}\int_{0}^{\frac{\pi}{2}}\frac{1}{24}\cos(3\,\phi)\,(1-\cos(\theta))^{(3/2)}\left(\frac{1}{8}+\frac{3}{8}\cos(\theta)+\frac{3}{8}\cos(\theta)^{2}+\frac{1}{8}\cos(\theta)^{3}\right)\sqrt{5040}\,\sin(\theta)$$

$$\left(-\frac{1}{\sqrt{1.0000001+\sin(\theta)\cos(\phi)}}-\frac{1}{\sqrt{1.0000001-\sin(\theta)\cos\left(\phi+\frac{\pi}{3}\right)}}\right)\Big/\left(\pi\,(\cos(\theta)+1)^{(3/2)}\right)\mathrm{d}\theta\,\mathrm{d}\phi$$

regularPart := -0.0707862141257640

```
>
```

now the remaining part, Vs

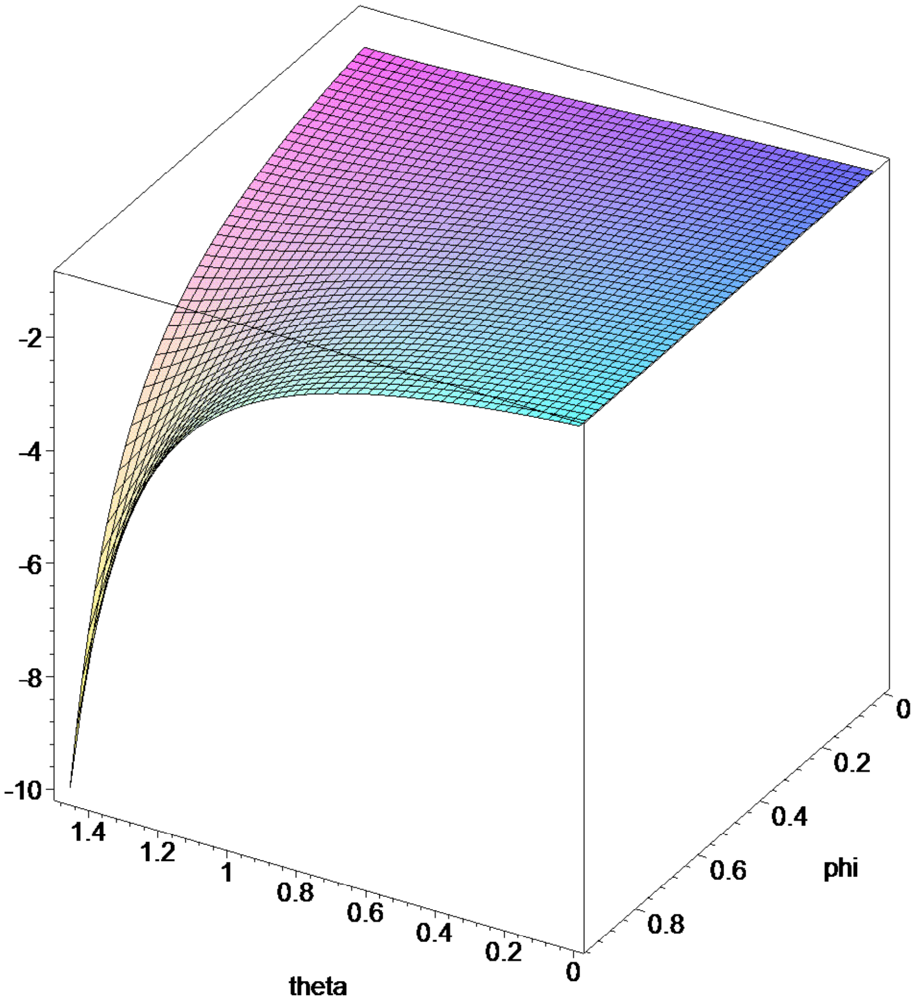
```
> assume (0 < epsilon); getassumptions(epsilon);
```

$$\{\varepsilon::\mathrm{RealRange}(\mathrm{Open}(0),\infty)\}$$

```
> #Ki/Vsi*
  Vs; eval(%, epsilon = 10^(-7));
  plot3d(%, theta=0..Pi/2 - 1e-1,phi=0..2*Pi/3/2 - 1e-1,  axes=boxed, orientation=[120,60]);
```

$$-\frac{1}{\sqrt{1+\varepsilon-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}$$

$$-\frac{1}{\sqrt{\frac{10000001}{10000000}-\sin(\theta)\sin\left(\phi+\frac{\pi}{6}\right)}}$$




```
> #Ki/Vsi*
  #Vs; #combine(%); simplify(%, size);
  'Int(Ki/Vsi * Vs, phi = 0 .. 1/3*Pi)'; value(%) assuming 0<theta, theta < Pi/2:
  convert(%, hypergeom):
  #simplify(%, size) assuming 0<theta, theta < Pi/2;
  collect(%,[EllipticE, EllipticF, EllipticK, hypergeom]):
  simplify(%, size) assuming 0<theta, theta < Pi/2:
  #Tryhard(%);
  #combine(%) assuming 0<theta, theta < Pi/2;
  #collect(%, EllipticE);
  U:=%;
```

$$\int_0^{\frac{\pi}{3}} \frac{\mathrm{Ki}\ \mathrm{Vs}}{\mathrm{Vsi}}\, d\phi$$

$$U := \frac{2}{15}\sqrt{35}\left(2\,(\varepsilon+1)\,(\cos(\theta)-1)\,(\cos(\theta)+1)\left(-\frac{17}{32}\sin(\theta)^2+(\varepsilon+1)^2\right)\mathrm{EllipticF}\left(\frac{\sqrt{3}}{2}, \sqrt{2}\,\sqrt{\frac{\sin(\theta)}{\varepsilon+1+\sin(\theta)}}\right)\right.$$

$$-\,2\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(\varepsilon+1+\sin(\theta))\left(-\frac{9}{32}\sin(\theta)^2+(\varepsilon+1)^2\right)\mathrm{EllipticE}\left(\frac{\sqrt{3}}{2}, \sqrt{2}\,\sqrt{\frac{\sin(\theta)}{\varepsilon+1+\sin(\theta)}}\right)$$

$$-\,(\varepsilon+1)\,(\cos(\theta)-1)\,(\cos(\theta)+1)\left(-\frac{17}{32}\sin(\theta)^2+(\varepsilon+1)^2\right)\pi\,\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}\right],[1],\frac{2\sin(\theta)}{\varepsilon+1+\sin(\theta)}\right)$$

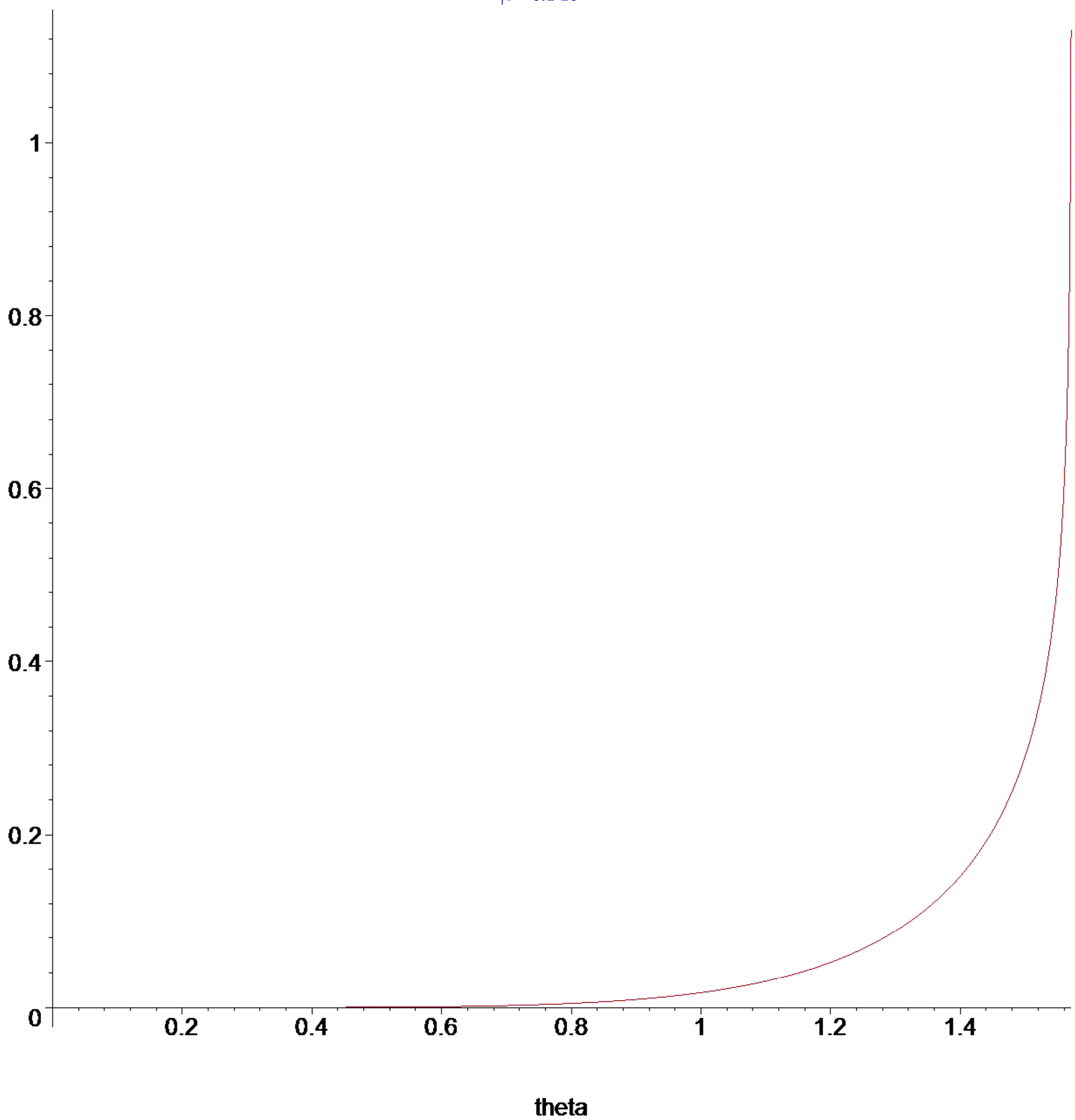
$$+\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(\varepsilon+1+\sin(\theta))\left(-\frac{9}{32}\sin(\theta)^2+(\varepsilon+1)^2\right)\pi\,\mathrm{hypergeom}\left(\left[\frac{-1}{2},\frac{1}{2}\right],[1],\frac{2\sin(\theta)}{\varepsilon+1+\sin(\theta)}\right)$$

$$-\frac{1}{4}\sqrt{\varepsilon+1+\sin(\theta)}\,\sqrt{12+12\,\varepsilon-6\sin(\theta)}\,\sin(\theta)^3\left(\varepsilon+\frac{3}{8}\sin(\theta)+1\right)\Big)\Big)\sqrt{1-\cos(\theta)}\,\sqrt{\cos(\theta)+1}\Big/\left(\sqrt{\varepsilon+1+\sin(\theta)}\,\pi\sin(\theta)^2\right)$$

Astonishing: a symbolic answer

```
> 'eval(U, epsilon=1e-7)';
  plot(%, theta = 0 .. Pi/2);
  eval(U, theta=Pi/2): eval(%, epsilon=1e-7): evalf(%);
```

$$U\Big|_{\varepsilon=0.1\ 10^{-6}}$$



1.13083845318858

```
> 'eval(U, epsilon=1e-7)';
  12* # we need 12 times by the range reduction in the above
  Int(%, theta = 0 .. Pi/2, method = _d01ajc);
  singularPart:=evalf(%);
```

$$U\Big|_{\varepsilon=0.1\ 10^{-6}}$$

$$12\ \mathrm{Int}\Bigg(\frac{2}{15}\sqrt{35}\,\Bigg(2.0000002\,(\cos(\theta)-1)\,(\cos(\theta)+1)\left(-\frac{17}{32}\sin(\theta)^2+1.00000020000001\right)\mathrm{EllipticF}\left(\frac{\sqrt{3}}{2},\sqrt{2}\sqrt{\frac{\sin(\theta)}{1.0000001+\sin(\theta)}}\right)$$
$$-2\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(1.0000001+\sin(\theta))\left(-\frac{9}{32}\sin(\theta)^2+1.00000020000001\right)\mathrm{EllipticE}\left(\frac{\sqrt{3}}{2},\sqrt{2}\sqrt{\frac{\sin(\theta)}{1.0000001+\sin(\theta)}}\right)$$
$$-1.0000001\,(\cos(\theta)-1)\,(\cos(\theta)+1)\left(-\frac{17}{32}\sin(\theta)^2+1.00000020000001\right)\pi\ \mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}\right],[1],\frac{2\sin(\theta)}{1.0000001+\sin(\theta)}\right)$$

$$+\left(\cos(\theta)-1\right)\left(\cos(\theta)+1\right)\left(1.0000001+\sin(\theta)\right)\left(-\frac{9}{32}\sin(\theta)^2+1.00000020000001\right)\pi\,\mathrm{hypergeom}\left(\left[\frac{-1}{2},\frac{1}{2}\right],[1],\frac{2\sin(\theta)}{1.0000001+\sin(\theta)}\right)$$

$$-\frac{1}{4}\sqrt{1.0000001+\sin(\theta)}\,\sqrt{12.0000012-6\sin(\theta)}\,\sin(\theta)^3\left(1.0000001+\frac{3}{8}\sin(\theta)\right)\Bigg)\Bigg)\sqrt{1-\cos(\theta)}\,\sqrt{\cos(\theta)+1}\;\Big/\;\left(\sqrt{1.0000001+\sin(\theta)}\,\pi\right.$$

$$\left.\sin(\theta)^2\right),\ \theta=0\,..\,\frac{\pi}{2},\ \mathrm{method}=\_d01ajc\Bigg)$$

singularPart := 0.965805647166259

```
>
>
>
>
> regularPart + singularPart;
```

0.895019433040495

```
>
> # does not work:
  #'Int(Ki/Vsi * (V2+V3), phi = 0 .. 1/3*Pi)';
  #value(%) assuming 0<theta, theta < Pi/2;
> # lost patients:
  #'Int(Ki/Vsi * (V2+V3), theta = 0 .. Pi/2)';
  #value(%) assuming 0<phi, phi < Pi/3;
>
```