To answer all of these questions, we must first deal with some procedural problems that come about from having to take derivatives of infinite series. For a finite series, such problems do not exist because from ordinary calculus, we know that the derivative of a sum equals the sum of the derivatives. However, when the sum is an infinite sum, the preceding may or may not be true. We now list some theorems, without proof, that will play a role in the verification procedure for solutions of the preceding problems over the rectangular domain $D = \{(x,t) \mid a < x < b, t > 0\}$.

Theorem 3.7.1 (Convergence of Derivatives) Consider the following infinite series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

If all the series terms $u_n(x)$ are differentiable on $I = [a, b]$ and if the series of differentiated terms

$$\sum_{n=0}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x} u_n(x) \right)$$

converges uniformly on I, then the series converges uniformly to the function $u(x)$ and the derivative of the series equals the series of the derivatives; that is,

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x) = \sum_{n=0}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x} u_n(x) \right)$$

for all x in I.

Theorem 3.7.2 (The Weierstrass M-test for Uniform Convergence) If the terms of the preceding series satisfy the condition

$$|u_n(x)| \leq M_n$$

for all n and for all x in $I = [a, b]$, where the M_n are constants (independent of x), and if the series of constants

$$\sum_{n=0}^{\infty} M_n$$

converges, then the series

$$\sum_{n=0}^{\infty} u_n(x)$$

converges uniformly for all x in the interval I .

We now apply the three-step verification procedure to verify the solution to the illustration problem in Section 3.5.

The homogeneous diffusion equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

The boundary conditions are type 1 at $x = 0$ and type 1 at $x = 1$

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0$$

The initial condition is

$$u_{x,0} = x(1-x)$$

From Section 3.5, the solution to the equation for $k = 1/10$ reads

$$u(x,t) = \sum_{n=1}^{\infty}\left(-\frac{4\left((-1)^n - 1\right)e^{-\frac{n^2\pi^2 t}{10}}\sin(n\pi x)}{n^3\pi^3}\right)$$

For the first step, we check to see if this solution satisfies the partial differential equation. Differentiating formally, once with respect to t and twice with respect to x, we get

$$\frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty}\frac{2\left((-1)^n - 1\right)e^{-\frac{n^2\pi^2 t}{10}}\sin(n\pi x)}{5n\pi}$$

and

$$\frac{\partial^2}{\partial x^2}u(x,t) = \sum_{n=1}^{\infty}\frac{4\left((-1)^n - 1\right)e^{-\frac{n^2\pi^2 t}{10}}\sin(n\pi x)}{n\pi}$$

It is obvious that for $k = 1/10$, both sides of the partial differential equation are satisfied; that is,

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

The preceding differentiations were done formally; that is, we wrote the derivative of the series as being the series of the derivatives. To verify the validity of such a move, we must use Theorems 3.7.1 and 3.7.2. The n-th term of both differentiated series given reads

$$\frac{2\left((-1)^n - 1\right)e^{-\frac{n^2\pi^2 t}{10}}\sin(n\pi x)}{5n\pi}$$

For x in the interval $I = [0, 1]$, the absolute value of the preceding term is less than or equal to the following term:

$$\frac{2e^{-\frac{n^2\pi^2 t}{10}} \left|\left((-1)^n - 1\right)\sin(n\pi x)\right|}{5n\pi} \leq \frac{e^{-\frac{n^2\pi^2 t}{10}}}{n}$$

Using the Weierstrass M-test on this inequality, in addition to using the ratio test on the following series

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{n^2\pi^2 t}{10}}}{n}$$

indicates the series converges for $t > 0$. Thus, since the series converges absolutely for all x in I, then, from Theorem 3.7.2, both of the differentiated series converge uniformly, and this justifies the formal operation of differentiation.

The second step in the verification procedure is to confirm that the boundary conditions are satisfied. Since the solution is a generalized Fourier series expansion in terms of the eigenfunctons, which satisfy the same boundary conditions (this is always the case for homogeneous boundary conditions), then the boundary conditions on the solution are, indeed, satisfied. Obviously, substituting $x = 0$ and $x = 1$ into the solution yields

$$u(0, t) = 0 \quad \text{and} \quad u(1, t) = 0$$

The third and final step in the verification procedure is to check whether the initial condition is satisfied. If we substitute $t = 0$ into the preceding solution, we get

$$u(x, 0) = \sum_{n=1}^{\infty} \left(-\frac{4\left((-1)^n - 1\right)\sin(n\pi x)}{n^3\pi^3} \right)$$

Since the initial condition function $f(x)$ is required to be piecewise continuous over the interval I, we see that the given series is the generalized Fourier series expansion of $f(x) = x(1 - x)$ in terms of the "complete" set of orthonormalized eigenfunctions

$$X_n(x) = \sqrt{2}\sin(n\pi x)$$

A plot of both the initial condition function $f(x)$ and the series representation of the solution $u(x, 0)$ is shown in Figure 3.7. The accuracy of the series representation is obvious.

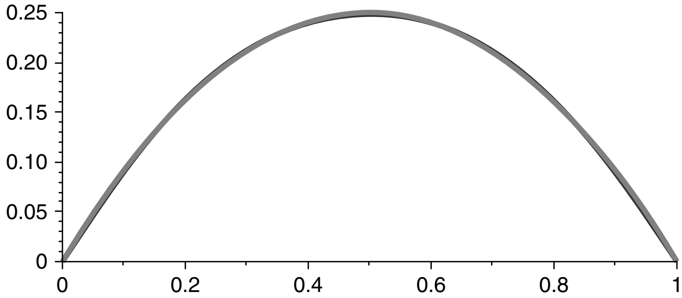

Figure 3.7

3.8 Diffusion Equation in the Cylindrical Coordinate System

The partial differential equation for diffusion or heat phenomena in the rectangular-cartesian coordinate system is presented in Section 3.2. The equivalent equation in the polar-cylindrical coordinate system for a circularly symmetric system, with spatially invariant thermal coefficients, is given as (see references for the conversion)

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r} + h(r,t)$$

In this equation, r is the coordinate radius of the system. There is no angle dependence here because we have assumed circular symmetry. Further, there is no z dependence because we will be considering problems that have no extension along the z-axis (thin plates).

As for the rectangular coordinate system, we can write the preceding equation in terms of the linear operator for the diffusion equation in rectangular coordinates as

$$L(u) = h(r,t)$$

where the diffusion operator in cylindrical coordinates with cylindrical symmetry is

$$L(u) = \frac{\partial}{\partial t}u(r,t) - \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

The homogeneous version of the diffusion equation can be written as

$$L(u) = 0$$

and this is generally written in the more familiar form

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

We seek solutions to this partial differential equation over the finite interval $I = \{r \,|\, a < r < b\}$ subject to the nonregular homogeneous boundary conditions

$$|u(a, t)| < \infty$$

and

$$\kappa_1 u(b, t) + \kappa_2 u_r(b, t) = 0$$

and the initial condition

$$u(r, 0) = f(r)$$

We now attempt to solve this partial differential equation using the method of separation of variables. We set

$$u(r, t) = R(r)T(t)$$

Substituting this into the preceding homogeneous partial differential equation, we get

$$R(r)\left(\frac{\mathrm{d}}{\mathrm{d}t}T(t)\right) = \frac{k\left(\frac{\mathrm{d}}{\mathrm{d}r}R(r) + r\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r)\right)\right)T(t)}{r}$$

Dividing both sides by the product solution yields

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}T(t)}{kT(t)} = \frac{\frac{\mathrm{d}}{\mathrm{d}r}R(r) + r\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r)\right)}{R(r)r}$$

Since the left-hand side of the preceding is an exclusive function of t and the right-hand side an exclusive function of r, and r and t are independent, then the only way we can ensure equality for all r and t is to set each side equal to a constant.

Doing so, we arrive at the following two ordinary differential equations in terms of the separation constant λ^2:

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t) + k\lambda^2 T(t) = 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r) + \frac{\frac{\mathrm{d}}{\mathrm{d}r}R(r)}{r} + \lambda^2 R(r) = 0$$

The preceding differential equation in t is an ordinary first-order linear equation for which we already have the solution from Chapter 1.

The second differential equation in the variable r is recognized from Section 1.10 as being an ordinary Bessel differential equation of order zero. The solution of this equation is the Bessel function of the first kind of order zero.

It was noted in Section 2.6 that, for the Bessel differential equation, the point $r = 0$ is a regular singular point of the differential equation. With appropriate boundary conditions over an interval that includes the origin, we obtain a "nonregular" (singular) type Sturm-Liouville eigenvalue problem whose eigenfunctions form an orthogonal set.

Similar to regular Sturm-Liouville problems over finite intervals, there exists an infinite number of eigenvalues that can be indexed by the positive integers n. The indexed eigenvalues and corresponding eigenfunctions are given, respectively, as

$$\lambda_n, R_n(r)$$

for $n = 0, 1, 2, 3, \ldots$.

The eigenfunctions form a "complete" set with respect to any piecewise smooth function over the finite interval $I = \{r \mid a < r < b\}$. In Section 2.6, we examined the nature of the orthogonality of the Bessel functions, and we showed the eigenfunctions to be orthogonal with respect to the weight function $w(r) = r$ over a finite interval I. Further, the eigenfunctions can be normalized and the corresponding statement of orthonormality reads

$$\int_a^b R_n(r) R_m(r) r \, \mathrm{d}r = \delta(n, m)$$

where the term on the right is the familiar Kronecker delta function.

Using arguments similar to those for the regular Sturm-Liouville problem, we can write our general solution to the partial differential equation as a superposition of the products of the solutions to each of the ordinary differential equations given earlier.

For the indexed values of λ, the solution to the preceding time-dependent equation is

$$T_n(t) = C(n) \mathrm{e}^{-k\lambda_n^2 t}$$

where the coefficients $C(n)$ are unknown arbitrary constants.

By the method of separation of variables, we arrive at an infinite number of indexed solutions $u_n(r, t)$ ($n = 0, 1, 2, 3, \ldots$) for the homogeneous diffusion partial differential equation, over a finite interval, given as

$$u_n(r, t) = R_n(r) C(n) \mathrm{e}^{-k\lambda_n^2 t}$$

Because the differential operator is linear, then any superposition of solutions to the homogeneous equation is also a solution. Thus, the general solution can be written as the infinite sum

$$u(r, t) = \sum_{n=0}^{\infty} R_n(r) C(n) \mathrm{e}^{-k\lambda_n^2 t}$$

We demonstrate the preceding concepts with an example diffusion problem in cylindrical coordinates.

DEMONSTRATION: We seek the temperature distribution $u(r, t)$ in a thin circularly symmetric plate over the finite interval $I = \{r \mid 0 < r < 1\}$ whose lateral surface is insulated, so there is no heat loss through the lateral surfaces. The periphery (edge) of the plate is at the fixed temperature of zero. The initial temperature distribution $f(r)$ is given below, and the diffusivity is $k = 1/20$.

SOLUTION: The homogeneous diffusion equation is

$$\frac{\partial}{\partial t}u(r,t) = \frac{\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)}{20r}$$

The boundary conditions are type 1 at $r = 1$, and we require the solution to be finite at the origin

$$|u(0,t)| < \infty \quad \text{and} \quad u(1,t) = 0$$

The initial condition is

$$u(r,0) = f(r)$$

From the method of separation of variables, we obtain the two ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t) + \frac{\lambda^2 T(t)}{20} = 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r) + \frac{\frac{\mathrm{d}}{\mathrm{d}r}R(r)}{r} + \lambda^2 R(r) = 0$$

We first consider the spatial differential equation in r. This is a Bessel-type differential equation of the first kind of order zero with boundary conditions

$$|R(0)| < \infty \quad \text{and} \quad R(1) = 0$$

This same problem was considered in Example 2.6.2 in Chapter 2. The allowed eigenvalues are the roots of the eigenvalue equation

$$J(0, \lambda_n) = 0$$

for $n = 1, 2, 3, \ldots$, and the corresponding orthonormal eigenfunctions are

$$R_n(r) = \frac{\sqrt{2}J(0, \lambda_n r)}{J(1, \lambda_n)}$$

where $J(0, \lambda_n)$ and $J(1, \lambda_n)$ are the Bessel functions of the first kind of order zero and one, respectively. The corresponding statement of orthonormality with respect to the weight function $w(r) = r$ over the interval I is

$$\int_0^1 \frac{2J(0, \lambda_n r)J(0, \lambda_m r)r}{J(1, \lambda_n)J(1, \lambda_m)} \mathrm{d}r = \delta(n, m)$$

We next consider the time-dependent differential equation. This is a first-order ordinary differential equation that we solved in Section 1.2. The solution for the allowed values of λ given earlier reads

$$T_n(t) = C(n)\mathrm{e}^{-\frac{\lambda_n^2 t}{20}}$$

Thus, the eigenfunction expansion for the solution to the problem reads

$$u(r, t) = \sum_{n=1}^{\infty} \frac{C(n)\mathrm{e}^{-\frac{\lambda_n^2 t}{20}} \sqrt{2}J(0, \lambda_n r)}{J(1, \lambda_n)}$$

The unknown coefficients $C(n)$, for $n = 1, 2, 3, \ldots$, are to be determined from the initial condition function imposed on the problem.

3.9 Initial Conditions for the Diffusion Equation in Cylindrical Coordinates

We now consider the initial conditions on the problem. If the initial condition temperature distribution is given as

$$u(r, 0) = f(r)$$

then substitution of this into the following general solution

$$u(r, t) = \sum_{n=0}^{\infty} R_n(r)C(n)\mathrm{e}^{-k\lambda_n^2 t}$$

at time $t = 0$ yields

$$f(r) = \sum_{n=0}^{\infty} R_n(r)C(n)$$

This equation is the Fourier-Bessel series expansion of the function $f(r)$, and the coefficients $C(n)$ are the Fourier coefficients.

As we did before for the generalized Fourier series expansion of a piecewise smooth function over the finite interval I, we can evaluate the coefficients $C(n)$ by taking the inner product of both sides of the preceding equation with the orthonormalized eigenfunctions with respect to the weight function $w(r) = r$. Assuming validity of the interchange between the summation and integration operations, we get

$$\int_a^b f(r)R_m(r)r\,\mathrm{d}r = \sum_{n=0}^{\infty} C(n)\left(\int_a^b R_n(r)R_m(r)r\,\mathrm{d}r\right)$$

Taking advantage of the statement of orthonormality, this equation reduces to

$$\int_a^b f(r)R_m(r)r\,\mathrm{d}r = \sum_{n=0}^{\infty} C(n)\delta(n,m)$$

Due to the mathematical character of the Kronecker delta function, only one term $(n = m)$ in the sum survives, and we get

$$C(m) = \int_a^b f(r)R_m(r)r\,\mathrm{d}r$$

Thus, we can write the final generalized solution to the diffusion equation in cylindrical coordinates in one dimension, subject to the given homogeneous boundary conditions and initial conditions, as

$$u(r,t) = \sum_{n=0}^{\infty} R_n(r)\mathrm{e}^{-k\lambda_n^2 t}\left(\int_a^b f(s)R_n(s)s\,\mathrm{d}s\right)$$

Again, all of the preceding operations are based on the assumption that the infinite series is uniformly convergent and the formal interchange between the operator and the summation is legitimate. It can be shown that if the initial condition function $f(r)$ is piecewise smooth and it satisfies the same boundary conditions as the eigenfunctions, then the preceding series is, indeed, uniformly convergent.

DEMONSTRATION: We now provide a demonstration of these concepts for the example problem given in Section 3.8 for the case where the initial temperature distribution is

$$f(r) = 1 - r^2$$

SOLUTION: The unknown Fourier coefficients are to be evaluated from the integral

$$C(n) = \int_0^1 \frac{(1-r^2)\sqrt{2}J(0,\lambda_n r)r}{J(1,\lambda_n)}\mathrm{d}r$$

Evaluation of this integral yields

$$C(n) = \frac{4\sqrt{2}}{\lambda_n^3}$$

for $n = 1, 2, 3, \ldots$. Thus, the final series solution to our problem reads

$$u(r, t) = \sum_{n=1}^{\infty} \frac{8e^{-\frac{\lambda_n^2 t}{20}} J(0, \lambda_n r)}{\lambda_n^3 J(1, \lambda_n)}$$

The detailed development of the solution of this problem along with the graphics are given in one of the Maple worksheet examples given later.

3.10 Example Diffusion Problems in Cylindrical Coordinates

We now consider several examples of partial differential equations for heat or diffusion phenomena under various homogeneous boundary conditions over finite intervals in the cylindrical coordinate system. We note that all the spatial ordinary differential equations in the cylindrical coordinate system are of the Bessel type and the solutions are Bessel functions of the first kind.

EXAMPLE 3.10.1: We seek the temperature distribution $u(r, t)$ in a thin circularly symmetric plate over the interval $I = \{r \mid 0 < r < 1\}$ whose lateral surface is insulated. The periphery (edge) of the plate is at the fixed temperature of zero. The initial temperature distribution $f(r)$ is given following, and the diffusivity is $k = 1/20$.

SOLUTION: The homogeneous diffusion equation is

$$\frac{\partial^2}{\partial t^2} u(r, t) = \frac{k\left(\frac{\partial}{\partial r} u(r, t) + r\left(\frac{\partial^2}{\partial r^2} u(r, t)\right)\right)}{r}$$

The boundary conditions are type 1 at $r = 1$, and we require a finite solution at $r = 0$.

$$|u(0, t)| < \infty \quad \text{and} \quad u(1, t) = 0$$

The initial condition is

$$u(r, 0) = 1 - r^2$$

Ordinary differential equations obtained from the method of separation of variables are

$$\frac{d}{dt} T(t) + k\lambda^2 T(t) = 0$$

and

$$\frac{d^2}{dr^2}R(r)+\frac{\frac{d}{dr}R(r)}{r}+\lambda^2 R(r)=0$$

Boundary conditions on the spatial equation are

$$|R(0)|<\infty \quad \text{and} \quad R(1)=0$$

Assignment of system parameters

```
> restart:with(plots):a:=0:b:=1:k:=1/20:
```

Allowed eigenvalues and orthonormal eigenfunctions are from Example 2.6.2. The eigenvalues are the roots of the eigenvalue equation

```
> BesselJ(0,lambda[n]*b)=0;
```

$$\text{BesselJ}(0,\lambda_n)=0 \tag{3.65}$$

for $n=1,2,3,\ldots$.

Orthonormal eigenfunctions

```
> R[n](r):=simplify(BesselJ(0,lambda[n]*r)/sqrt(int(BesselJ(0,lambda[n]*r)^2*r,r=a..b)));
```

$$R_n(r):=\frac{\text{BesselJ}(0,\lambda_n r)\sqrt{2}}{\sqrt{\text{BesselJ}(0,\lambda_n)^2+\text{BesselJ}(1,\lambda_n)^2}} \tag{3.66}$$

Substitution of the eigenvalue equation simplifies the preceding equation

```
> R[n](r):=radsimp(subs(BesselJ(0,lambda[n])=0,R[n](r)));R[m](r):=subs(n=m,R[n](r)):
```

$$R_n(r):=\frac{\text{BesselJ}(0,\lambda_n r)\sqrt{2}}{\text{BesselJ}(1,\lambda_n)} \tag{3.67}$$

Statement of orthonormality with respect to the weight function $w(r)=r$

```
> w(r):=r:Int(R[n](r)*R[m](r)*w(r),r=a...b)=delta(n,m);
```

$$\int_0^1 \frac{2\text{BesselJ}(0,\lambda_n r)\text{BesselJ}(0,\lambda_m r)r}{\text{BesselJ}(1,\lambda_n)\text{BesselJ}(1,\lambda_m)}dr=\delta(n,m) \tag{3.68}$$

Time-dependent solution

```
> T[n](t):=C(n)*exp(−k*lambda[n]^2*t);u[n](r,t):=T[n](t)*R[n](r):
```

$$T_n(t):=C(n)e^{-\frac{1}{20}\lambda_n^2 t} \tag{3.69}$$

Generalized series terms

> u[n](r,t):=T[n](t)*R[n](r);

$$u_n(r,t) := \frac{C(n)\mathrm{e}^{-\frac{1}{20}\lambda_n^2 t}\mathrm{BesselJ}(0,\lambda_n r)\sqrt{2}}{\mathrm{BesselJ}(1,\lambda_n)} \tag{3.70}$$

Eigenfunction expansion

> u(r,t):=Sum(u[n](r,t),n=1..infinity);

$$u(r,t) := \sum_{n=1}^{\infty} \frac{C(n)\mathrm{e}^{-\frac{1}{20}\lambda_n^2 t}\mathrm{BesselJ}(0,\lambda_n r)\sqrt{2}}{\mathrm{BesselJ}(1,\lambda_n)} \tag{3.71}$$

Evaluation of Fourier coefficients for the specific initial condition

> f(r):=1−r^2;

$$f(r) := 1 - r^2 \tag{3.72}$$

> C(n):=Int(f(r)*R[n](r)*w(r),r=a..b);

$$C(n) := \int_0^1 \frac{(1-r^2)\mathrm{BesselJ}(0,\lambda_n r)\sqrt{2}r}{\mathrm{BesselJ}(1,\lambda_n)}\mathrm{d}r \tag{3.73}$$

Substitution of the eigenvalue equation simplifies the preceding equation

> C(n):=simplify(subs(BesselJ(0,lambda[n])=0,value(%)));u[n](r,t):=eval(T[n](t)*R[n](r)):

$$C(n) := \frac{4\sqrt{2}}{\lambda_n^3} \tag{3.74}$$

Generalized series terms

> u[n](r,t):=eval(T[n](t)*R[n](r));

$$u_n(r,t) := \frac{8\mathrm{e}^{-\frac{1}{20}\lambda_n^2 t}\mathrm{BesselJ}(0,\lambda_n r)}{\lambda_n^3\mathrm{BesselJ}(1,\lambda_n)} \tag{3.75}$$

Series solution

> u(r,t):=Sum(u[n](r,t),n=1..infinity);

$$u(r,t) := \sum_{n=1}^{\infty} \frac{8\mathrm{e}^{-\frac{1}{20}\lambda_n^2 t}\mathrm{BesselJ}(0,\lambda_n r)}{\lambda_n^3\mathrm{BesselJ}(1,\lambda_n)} \tag{3.76}$$

Evaluation of the eigenvalues from the roots of the eigenvalue equation yields

> BesselJ(0,lambda[n]*b)=0;

$$\text{BesselJ}(0, \lambda_n) = 0 \tag{3.77}$$

> plot(BesselJ(0,v),v=0..20,thickness=10);

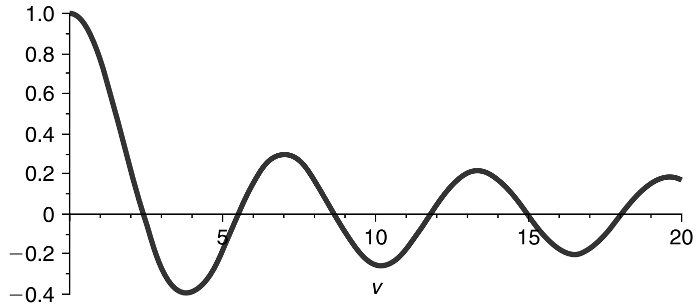

Figure 3.8

If we set $v = \lambda b$, then the eigenvalues are found from the intersection points of the curve with the v-axis shown in Figure 3.8. We evaluate a few of these eigenvalues using the Maple fsolve command:

> lambda[1]:=(1/b)*fsolve(BesselJ(0,v)=0,v=0..3);

$$\lambda_1 := 2.404825558 \tag{3.78}$$

> lambda[2]:=(1/b)*fsolve(BesselJ(0,v)=0,v=3..6);

$$\lambda_2 := 5.520078110 \tag{3.79}$$

> lambda[3]:=(1/b)*fsolve(BesselJ(0,v)=0,v=6..9);

$$\lambda_3 := 8.653727913 \tag{3.80}$$

First few terms in sum

> u(r,t):=eval(sum(u[n](r,t),n=1..3)):

ANIMATION

> animate(u(r,t),r=a..b,t=0..5,thickness=3);

The preceding animation command illustrates the spatial-time-dependent solution for $u(r, t)$. The animation sequence shown in Figure 3.9 shows snapshots at times $t = 0, 1, 2, 3, 4,$ and 5.

ANIMATION SEQUENCE

```
> u(r,0):=subs(t=0,u(r,t)):u(r,1):=subs(t=1,u(r,t)):
> u(r,2):=subs(t=2,u(r,t)):u(r,3):=subs(t=3,u(r,t)):
> u(r,4):=subs(t=4,u(r,t)):u(r,5):=subs(t=5,u(r,t)):
> plot({u(r,0),u(r,1),u(r,2),u(r,3),u(r,4),u(r,5)},r=a..b,thickness=10);
```

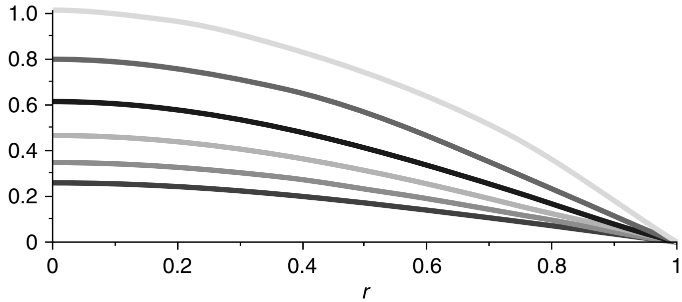

Figure 3.9

THREE-DIMENSIONAL ANIMATION

```
> u(x,y,t):=eval(subs(r=sqrt(x^2+y^2),u(r,t))):
> u(x,y,t):=(u(x,y,t))*Heaviside(1−sqrt(x^2+y^2)):
> animate3d(u(x,y,t),x=−b..b,y=−b..b,t=0..5,axes=framed,thickness=1);
```

EXAMPLE 3.10.2: We again seek the temperature distribution $u(r, t)$ in a thin circularly symmetric plate over the interval $I = \{r \mid 0 < r < 1\}$. The lateral surface and the periphery (edge) of the plate are insulated. The initial temperature distribution $f(r)$ is given below, and the diffusivity is $k = 1/50$.

SOLUTION: The homogeneous diffusion equation is

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

The boundary conditions are type 2 at $r = 1$, and we require a finite solution at $r = 0$.

$$|u(0,t)| < \infty \quad \text{and} \quad u_r(1,t) = 0$$

The initial condition is

$$u(r,0) = 1 - r^2$$

Ordinary differential equations obtained from the method of separation of variables are

$$\frac{d}{dt}T(t) + k\lambda^2 T(t) = 0$$

and

$$\frac{d^2}{dr^2}R(r) + \frac{\frac{d}{dr}R(r)}{r} + \lambda^2 R(r) = 0$$

Boundary conditions on the spatial equation

$$|R(0)| < \infty \quad \text{and} \quad R_r(1) = 0$$

Assignment of system parameters

```
> restart:with(plots):a:=0:b:=1:k:=1/50:
```

Allowed eigenvalues and orthonormal eigenfunctions are obtained from Example 2.6.3.

```
> lambda[0]:=0;
```

$$\lambda_0 := 0 \tag{3.81}$$

for $n = 0$.

Orthonormal eigenfunction

```
> R[0](r):=sqrt(2)/b;
```

$$R_0(r) := \sqrt{2} \tag{3.82}$$

For $n = 1, 2, 3, \ldots$, the eigenvalues are the roots of the eigenvalue equation

```
> subs(r=b,diff(BesselJ(0,lambda[n]*r),r))=0;
```

$$-\text{BesselJ}(1, \lambda_n)\lambda_n = 0 \tag{3.83}$$

Orthonormal eigenfunctions

```
> R[n](r):=simplify(BesselJ(0,lambda[n]*r)/sqrt(int(BesselJ(0,lambda[n]*r)^2*r,r=a..b)));
```

$$R_n(r) := \frac{\text{BesselJ}(0, \lambda_n r)\sqrt{2}}{\sqrt{\text{BesselJ}(0, \lambda_n)^2 + \text{BesselJ}(1, \lambda_n)^2}} \tag{3.84}$$

Substitution of the eigenvalue equation simplifies the preceding equation

```
> R[n](r):=radsimp(subs(BesselJ(1,lambda[n])=0,R[n](r)));R[m](r):=subs(n=m,R[n](r)):
```

$$R_n(r) := \frac{\text{BesselJ}(0, \lambda_n r)\sqrt{2}}{\text{BesselJ}(0, \lambda_n)} \tag{3.85}$$

Statement of orthonormality with respect to the weight function $w(r) = r$

> w(r):=r:Int(R[n](r)*R[m](r)*w(r),r=a...b)=delta(n,m);

$$\int_0^1 \frac{2\,\mathrm{BesselJ}(0, \lambda_n r)\mathrm{BesselJ}(0, \lambda_m r) r}{\mathrm{BesselJ}(0, \lambda_n)\mathrm{BesselJ}(0, \lambda_m)} \mathrm{d}r = \delta(n, m) \tag{3.86}$$

Time-dependent solution for $n = 1, 2, 3, \ldots$ reads

> T[n](t):=C(n)*exp(−k*lambda[n]^2*t);u[n](r,t):=T[n](t)*R[n](r);

$$T_n(t) := C(n)\mathrm{e}^{-\frac{1}{50}\lambda_n^2 t} \tag{3.87}$$

Generalized series terms

> u[n](r,t):=T[n](t)*R[n](r);

$$u_n(r, t) := \frac{C(n)\mathrm{e}^{-\frac{1}{50}\lambda_n^2 t}\mathrm{BesselJ}(0, \lambda_n r)\sqrt{2}}{\mathrm{BesselJ}(0, \lambda_n)} \tag{3.88}$$

and for $n = 0$,

> T[0](t):=C(0);u[0](r,t):=T[0](t)*R[0](r):

$$T_0(t) := C(0) \tag{3.89}$$

Eigenfunction expansion

> u(r,t):=u[0](r,t)+Sum(u[n](r,t),n=1..infinity);

$$u(r, t) := C(0)\sqrt{2} + \sum_{n=1}^{\infty} \frac{C(n)\mathrm{e}^{-\frac{1}{50}\lambda_n^2 t}\mathrm{BesselJ}(0, \lambda_n r)\sqrt{2}}{\mathrm{BesselJ}(0, \lambda_n)} \tag{3.90}$$

Evaluation of Fourier coefficients from the given specific initial condition

> f(r):=1−r^2;

$$f(r) := 1 - r^2 \tag{3.91}$$

yields, for $n = 0$,

> C(0):=eval(Int(f(r)*R[0](r)*r,r=a..b));

$$C(0) := \int_0^1 \left(1 - r^2\right)\sqrt{2} r\, \mathrm{d}r \tag{3.92}$$

```
> C(0):=value(%);u[0](r,t):=eval(T[0](t)*R[0](r)):
```

$$C(0) := \frac{1}{4}\sqrt{2} \tag{3.93}$$

and for $n = 1, 2, 3, \ldots,$

```
> C(n):=Int(f(r)*R[n](r)*r,r=a..b);
```

$$C(n) := \int_0^1 \frac{\left(1-r^2\right)\mathrm{BesselJ}(0, \lambda_n r)\sqrt{2}r}{\mathrm{BesselJ}(0, \lambda_n)}\mathrm{d}r \tag{3.94}$$

Substitution of the eigenvalue equation simplifies the preceding equation

```
> C(n):=radsimp(subs(BesselJ(1,lambda[n]*b)=0,value(%)));
```

$$C(n) := -\frac{2\sqrt{2}}{\lambda_n^2} \tag{3.95}$$

Generalized series terms

```
> u[n](r,t):=eval(T[n](t)*R[n](r));
```

$$u_n(r, t) := -\frac{4\,\mathrm{e}^{-\frac{1}{50}\lambda_n^2 t}\mathrm{BesselJ}(0, \lambda_n r)}{\lambda_n^2\mathrm{BesselJ}(0, \lambda_n)} \tag{3.96}$$

Series solution

```
> u(r,t):=u[0](r,t)+Sum(u[n](r,t),n=1..infinity);
```

$$u(r, t) := \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{4\,\mathrm{e}^{-\frac{1}{50}\lambda_n^2 t}\mathrm{BesselJ}(0, \lambda_n r)}{\lambda_n^2\mathrm{BesselJ}(0, \lambda_n)}\right) \tag{3.97}$$

Evaluation of the eigenvalues from the roots of the eigenvalue equation yields

```
> BesselJ(1,lambda[n]*b)=0;
```

$$\mathrm{BesselJ}(1, \lambda_n) = 0 \tag{3.98}$$

```
> plot(BesselJ(1,v),v=0..20,thickness=10);
```

If we set $v = \lambda b$, then the eigenvalues are found from the intersection points of the curve with the v-axis shown in Figure 3.10. We evaluate a few of these eigenvalues using the Maple fsolve command:

```
> lambda[1]:=(1/b)*fsolve(BesselJ(1,v)=0,v=1..4);
```

$$\lambda_1 := 3.831705970 \tag{3.99}$$

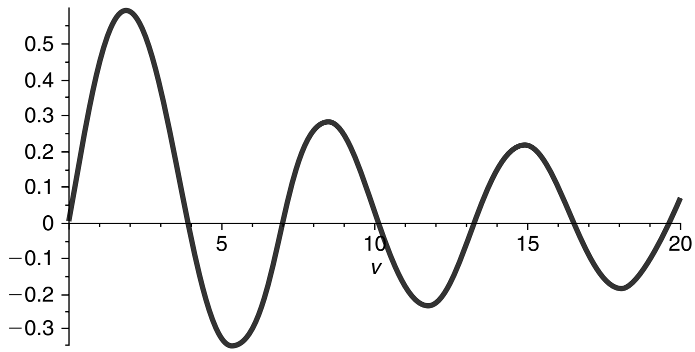

Figure 3.10

```
> lambda[2]:=(1/b)*fsolve(BesselJ(1,v)=0,v=4..8);
```

$$\lambda_2 := 7.015586670 \tag{3.100}$$

```
> lambda[3]:=(1/b)*fsolve(BesselJ(1,v)=0,v=8..12);
```

$$\lambda_3 := 10.17346814 \tag{3.101}$$

First few terms in the sum

```
> u(r,t):=u[0](r,t)+eval(sum(u[n](r,t),n=1..1)):
```

ANIMATION

```
> animate(u(r,t),r=a..b,t=0..5,thickness=3);
```

The preceding animation command illustrates the spatial-time-dependent solution for $u(r, t)$. The animation sequence shown in Figure 3.11 shows snapshots at times $t = 0$, 1, 2, 3, 4, and 5.

ANIMATION SEQUENCE

```
> u(r,0):=subs(t=0,u(r,t)):u(r,1):=subs(t=1,u(r,t)):
> u(r,2):=subs(t=2,u(r,t)):u(r,3):=subs(t=3,u(r,t)):
> u(r,4):=subs(t=4,u(r,t)):u(r,5):=subs(t=5,u(r,t)):
> plot({u(r,0),u(r,1),u(r,2),u(r,3),u(r,4),u(r,5)},r=a..b,thickness=10);
```

THREE-DIMENSIONAL ANIMATION

```
> u(x,y,t):=eval(subs(r=sqrt(x^2+y^2),u(r,t))):
> u(x,y,t):=(u(x,y,t))*Heaviside(1−sqrt(x^2+y^2)):
> animate3d(u(x,y,t),x=−b..b,y=−b..b,t=0..5,axes=framed,thickness=1);
```

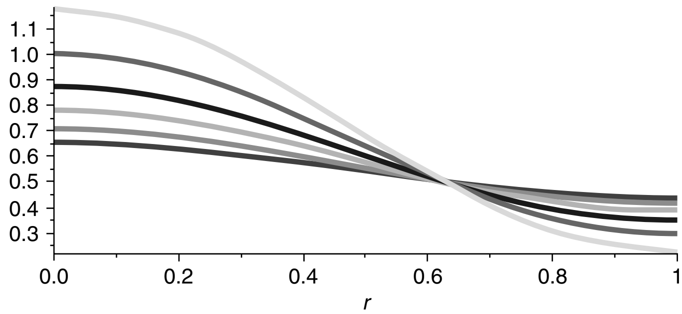

Figure 3.11

Chapter Summary

Nonhomogeneous one-dimensional diffusion equation in rectangular coordinates

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) + h(x,t)$$

Linear diffusion operator of one dimension in rectangular coordinates

$$L(u) = \frac{\partial}{\partial t}u(x,t) - k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

Method of separation of variables solution

$$u(x,t) = X(x)T(t)$$

Eigenfunction expansion solution for rectangular coordinates

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)C(n)\mathrm{e}^{-k\lambda_n t}$$

Initial condition Fourier coefficients for rectangular coordinates

$$C(n) = \int_a^b f(x)X_n(x)\,\mathrm{d}x$$

Nonhomogeneous one-dimensional diffusion equation in cylindrical coordinates

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r} + h(r,t)$$

Linear diffusion operator of one dimension in cylindrical coordinates

$$L(u) = \frac{\partial}{\partial t}u(r,t) - \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

Method of separation of variables solution

$$u(r,t) = R(r)T(t)$$

Eigenfunction expansion solution in cylindrical coordinates

$$u(r,t) = \sum_{n=0}^{\infty} R_n(r)C(n)\mathrm{e}^{-k\lambda_n^2 t}$$

Initial condition Fourier coefficients for cylindrical coordinates

$$C(n) = \int_a^b f(r)R_n(r)r\,\mathrm{d}r$$

We have examined partial differential equations describing diffusion or heat diffusion phenomena in a single spatial dimension for both the rectangular and the cylindrical coordinate systems. We examine these same partial differential equations in steady-state and higher-dimensional systems later.

Exercises

We now consider exercise problems dealing with diffusion or heat equations with homogeneous boundary conditions in both the rectangular and the cylindrical coordinate systems. Use the method of separation of variables and eigenfunction expansions to evaluate the solutions.

3.1. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

with $k = 1/10$, initial condition $u(x,0) = f(x)$ given following, and boundary conditions

$$u(0,t) = 0, u(1,t) = 0$$

that is, the left end of the rod is at the fixed temperature zero and the right end is at the fixed temperature zero. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = 1$$
$$f2(x) = x$$
$$f3(x) = x(1 - x) \tag{3.102}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.2. Use the three-step verification procedure in Exercise 3.1 for the case $u(x, 0) = f3(x)$.

3.3. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x, t) = k\left(\frac{\partial^2}{\partial x^2}u(x, t)\right)$$

with $k = 1/10$, initial condition $u(x, 0) = f(x)$ given later, and boundary conditions

$$u(0, t) = 0, u_x(1, t) = 0$$

that is, the left end of the rod is at the fixed temperature zero and the right end is insulated. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x^2$$
$$f2(x) = x$$
$$f3(x) = x\left(1 - \frac{x}{2}\right)$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.4. Use the three-step verification procedure in Exercise 3.3 for the case $u(x, 0) = f3(x)$.

3.5. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x, t) = k\left(\frac{\partial^2}{\partial x^2}u(x, t)\right)$$

with $k = 1/10$, initial condition $u(x, 0) = f(x)$ given later, and boundary conditions

$$u_x(0, t) = 0, u(1, t) = 0$$

that is, the left end of the rod is insulated and the right end is at the fixed temperature zero. Evaluate the eigenvalues and the corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x$$
$$f2(x) = 1$$
$$f3(x) = 1 - x^2$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.6. Use the three-step verification procedure in Exercise 3.5 for the case $u(x, 0) = f3(x)$.

3.7. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x, t) = k\left(\frac{\partial^2}{\partial x^2}u(x, t)\right)$$

with $k = 1/10$, initial condition $u(x, 0) = f(x)$ given later, and boundary conditions

$$u_x(0, t) = 0, u_x(1, t) = 0$$

that is, the left end of the rod is insulated and the right end is insulated. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x$$
$$f2(x) = 1 - x^2$$
$$f3(x) = x^2\left(1 - \frac{2x}{3}\right)$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.8. Use the three-step verification procedure in Exercise 3.7 for the case $u(x, 0) = f3(x)$.

3.9. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

with $k = 1/10$, initial condition $u(x,0) = f(x)$ given later, and boundary conditions

$$u(0,t) = 0, u(1,t) + u_x(1,t) = 0$$

that is, the left end of the rod is at a fixed temperature zero, and the right end is losing heat by convection into a zero temperature surrounding. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = 1$$

$$f2(x) = x$$

$$f3(x) = x\left(1 - \frac{2x}{3}\right)$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.10. Use the three-step verification procedure in Exercise 3.9 for the case $u(x,0) = f3(x)$.

3.11. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

with $k = 1/10$, initial condition $u(x,0) = f(x)$ given later, and boundary conditions

$$u(0,t) - u_x(0,t) = 0, u(1,t) = 0$$

that is, the left end of the rod is losing heat by convection into a zero temperature surrounding and the right end is at a fixed temperature zero. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = 1$$

$$f2(x) = 1 - x$$

$$f3(x) = -\frac{2x^2}{3} + \frac{x}{3} + \frac{1}{3}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.12. Use the three-step verification procedure in Exercise 3.11 for the case $u(x, 0) = f3(x)$.

3.13. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x, t) = k\left(\frac{\partial^2}{\partial x^2}u(x, t)\right)$$

with $k = 1/10$, initial condition $u(x, 0) = f(x)$ given later, and boundary conditions

$$u_x(0, t) = 0, u(1, t) + u_x(1, t) = 0$$

that is, the left end of the rod is insulated and the right end is losing heat by convection into a zero temperature surrounding. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x$$

$$f2(x) = 1$$

$$f3(x) = 1 - \frac{x^2}{3}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.14. Use the three-step verification procedure in Exercise 3.13 for the case $u(x, 0) = f3(x)$.

3.15. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x, t) = k\left(\frac{\partial^2}{\partial x^2}u(x, t)\right)$$

with $k = 1/10$, initial condition $u(x, 0) = f(x)$ given later, and boundary conditions

$$u(0, t) - u_x(0, t) = 0, u_x(1, t) = 0$$

that is, the left end of the rod is losing heat by convection into a zero temperature surrounding and the right end is insulated. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x$$
$$f2(x) = 1$$
$$f3(x) = 1 - \frac{(x-1)^2}{3}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.16. Use the three-step verification procedure in Exercise 3.15 for the case $u(x, 0) = f3(x)$.

3.17. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is not insulated. The rod is experiencing a heat loss proportional to the difference between the rod temperature and the surrounding temperature at zero degrees. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) - \beta u(x,t)$$

with $k = 1/10$, $\beta = 1/4$, initial condition $u(x, 0) = f(x)$ given later, and boundary conditions

$$u(0,t) = 0, u(1,t) = 0$$

that is, the left end of the rod is at the fixed temperature zero and the right end is at the fixed temperature zero. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = 1$$
$$f2(x) = x$$
$$f3(x) = x(1-x)$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.18. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is not insulated. The rod is experiencing a heat loss proportional to the difference between the rod temperature and the surrounding temperature at zero degrees. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) - \beta u(x,t)$$

with $k = 1/10$, $\beta = 1/4$, initial condition $u(x,0) = f(x)$ given later, and boundary conditions

$$u_x(0,t) = 0, \; u(1,t) = 0$$

that is, the left end of the rod is insulated and the right end is at a fixed temperature zero. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x$$

$$f2(x) = 1$$

$$f3(x) = 1 - x^2$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.19. Consider the temperature distribution in a thin rod over the interval $I = \{x \mid 0 < x < 1\}$ whose lateral surface is not insulated. The rod is experiencing a heat loss proportional to the difference between the rod temperature and the surrounding temperature at zero degrees. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(x,t) = k\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) - \beta u(x,t)$$

with $k = 1/10$, $\beta = 1/4$, initial condition $u(x,0) = f(x)$ given later, and boundary conditions

$$u_x(0,t) = 0, u(1,t) + u_x(1,t) = 0$$

that is, the left end of the rod is insulated and the right end is losing heat by convection into a zero temperature surrounding. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(x) = x$$

$$f2(x) = 1$$

$$f3(x) = 1 - \frac{x^2}{3}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.20. Consider the temperature distribution in a thin circularly symmetric plate over the interval $I = \{r \mid 0 < r < 1\}$. The lateral surface of the plate is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

with $k = 1/10$, initial condition $u(r,0) = f(r)$ given later, and boundary conditions

$$|u(0,t)| < \infty, u(1,t) = 0$$

that is, the center of the plate has a finite temperature and the periphery is at a fixed temperature zero. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(r) = r^2$$

$$f2(r) = 1$$

$$f3(r) = 1 - r^2$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.21. Consider the temperature distribution in a thin circularly symmetric plate over the interval $I = \{r \mid 0 < r < 1\}$. The lateral surface of the plate is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

with $k = 1/10$, initial condition $u(r,0) = f(r)$ given later, and boundary conditions

$$|u(0,t)| < \infty, u_r(1,t) = 0$$

that is, the center of the plate has a finite temperature and the periphery is insulated. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(r) = r^2$$
$$f2(r) = 1$$
$$f3(r) = r^2 - \frac{r^4}{2}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

3.22. Consider the temperature distribution in a thin circularly symmetric plate over the interval $I = \{r \mid 0 < r < 1\}$. The lateral surface of the plate is insulated. The homogeneous partial differential equation reads

$$\frac{\partial}{\partial t}u(r,t) = \frac{k\left(\frac{\partial}{\partial r}u(r,t) + r\left(\frac{\partial^2}{\partial r^2}u(r,t)\right)\right)}{r}$$

with $k = 1/10$, initial condition $u(r, 0) = f(r)$ given later, and boundary conditions

$$|u(0,t)| < \infty, u_r(1,t) + u(1,t) = 0$$

that is, the center of the plate has a finite temperature and the periphery is losing heat by convection into a zero temperature surrounding. Evaluate the eigenvalues and corresponding orthonormalized eigenfunctions, and write the general solution for each of these three initial condition functions:

$$f1(r) = r^2$$
$$f2(r) = 1$$
$$f3(r) = 1 - \frac{r^2}{3}$$

Generate the animated solution for each case, and plot the animated sequence for $0 < t < 5$. In the animated sequence, take note of the adherence of the solution to the boundary conditions.

Significance of Thermal Diffusivity

The coefficient k in the heat equation is called the “thermal diffusivity” of the medium under consideration. This constant is equal to

$$k = \frac{K}{c\rho}$$

where c is the specific heat of the medium, ρ is the mass density, and K is the thermal conductivity of the medium. In a uniform, isotropic medium, these terms are all constants. By convention, we say that heat is a diffusion process whereby heat flows from high-temperature regions to low-temperature regions similar to how salt in a water solution diffuses from high-concentration regions to low-concentration regions. The magnitude of the thermal diffusivity k is an indication of the ability of the medium to conduct heat from one region to another. Thus, large values of k provide for rapid transfers and small values of k provide for slow transfers of heat within the medium.

In the following, we investigate the significance of the change in magnitude of the diffusivity k by noting its effect on the solution of a problem. In Exercises 3.23 through 3.28, multiply the diffusivity by the given factor, and develop the solution for the given initial condition function $f3(x)$. Develop the animated solution and take particular note of the change in the time dependence of the solution due to the increased magnitude of the diffusivity.

3.23. In Exercise 3.1, multiply the diffusivity k by a factor of 5 and solve.

3.24. In Exercise 3.5, multiply the diffusivity k by a factor of 10 and solve.

3.25. In Exercise 3.9, multiply the diffusivity k by a factor of 5 and solve.

3.26. In Exercise 3.17, multiply the diffusivity k by a factor of 10 and solve.

3.27. In Exercise 3.20, multiply the diffusivity k by a factor of 5 and solve.

3.28. In Exercise 3.22, multiply the diffusivity k by a factor of 10 and solve.

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CHAPTER 4

The Wave Partial Differential Equation

4.1 Introduction

We begin by examining types of partial differential equations that exhibit wave phenomena. We see wave-type partial differential equations in many areas of engineering and physics, including acoustics, electromagnetic theory, quantum mechanics, and the study of the transmission of longitudinal and transverse disturbances in solids and liquids.

Similar to the partial differential equations that are descriptive of heat and diffusion phenomena, the wave partial differential equations that we examine here are also linear in that the partial differential operator L obeys the definition characteristics of a linear operator defined in Section 3.1.

4.2 One-Dimensional Wave Operator in Rectangular Coordinates

Wave phenomena in one dimension can be described by the following partial differential equation in the rectangular coordinate system:

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) - \gamma\left(\frac{\partial}{\partial t}u(x,t)\right) + h(x,t)$$

In the preceding, $u(x,t)$ denotes the spatial-time-dependent wave amplitude, c denotes the wave speed, γ denotes the damping coefficient of the medium, and $h(x,t)$ denotes the presence of any external applied forces acting on the system. If the medium is uniform, then all the preceding coefficients are spatially invariant, and we can write them as constants. The wave speed c is generally proportional to the square root of the quotient of the tension in the system and the inertia of the system (see Exercises 4.19 through 4.25).