Sample manuscript for solve same equations:

Shooting method

$$g'''' + (2/\xi)g''' - (1/\xi^2)g'' + (1/\xi^3)g' - \lambda g - [9(1-\nu^2)/\xi]\alpha(g'f)' = 0, \tag{8}$$

$$f'' + (1/\xi)f' - (1/\xi^2)f + (1/2\xi)(g')^2 = 0, \tag{9}$$

where the prime as the superscipt denotes differentiation, i.e., $f' = df/d\xi$, and where $\lambda = \omega^2$ and $\alpha = (Aa/h)^2$ are additional dimensionless parameters associated with the radian frequency and amplitude, respectively.

Similarly the boundary conditions of equations (6) are transformed to the non-dimensional form given in equations $(1/a, b)$ to follow. In addition to the boundary conditions, a normal relationship is imposed on the system: i.e.,

$$g|_{\xi=R} = 1. \tag{10}$$

The final non-dimensional normal and boundary conditions are summarized as follows: (a) for an immovable hinge,

$$\begin{aligned} &\text{at } \xi = 1, g = 0, \quad g'' + (\nu/\xi)g' = 0, \quad f' - \nu f = 0; \\ &\text{at } \xi = R, g = 1, \quad g' = 0, \quad f' - \nu f = 0, \quad g''' + (1/\xi)g'' - (\xi/2)\gamma\lambda g = 0; \end{aligned} \tag{11a}$$

(b) for a movable hinge,

$$\begin{aligned} &\text{at } \xi = 1, g = 0, \quad g'' + (\nu/\xi)g' = 0, \quad f/\xi = 0; \\ &\text{at } \xi = R, g = 1, \quad g' = 0, \quad f' - \nu f = 0, \quad g''' + (1/\xi)g'' - (\xi/2)\gamma\lambda g = 0. \end{aligned} \tag{11b}$$

Equations (8) and (9) are rewritten as six first-order ordinary differential equations as follows:

$$d\bar{Y}/d\xi = \bar{H}(\xi, Y; \alpha, \lambda, \gamma), \qquad R \leqslant \xi \leqslant 1, \tag{12a}$$

where

$$\bar{Y}(\xi) = [g, g', g'', g''', f, f']^{\mathrm{T}} = [y_1, y_2, y_3, y_4, y_5, y_6]^{\mathrm{T}}$$

and

$$\begin{aligned} \bar{H} = [&y_2, y_3, y_4, -(2/\xi)y_4 + (1/\xi^2)y_3 - (1/\xi^3)y_2 + \lambda y_1 \\ &+ \{(9(1-\nu)/\xi\}\alpha(y_2y_6 + y_3y_5), y_6, -(1/\xi)y_6 + (1/\xi^2)y_5 - (1/2\xi)y_2^2]^{\mathrm{T}}. \end{aligned}$$

The boundary and normalization conditions (10) and (11) are written in the matrix forms, respectively, as

$$M\bar{Y}(R) = [1, 0, 0, 0]^{\mathrm{T}}, \qquad N\bar{Y}(1) = [0, 0, 0]^{\mathrm{T}} \tag{12b, c}$$

where M and N are (4×6) and (3×6) coefficient matrices respectively. In Table 1 two sets of boundary matrices for M and N are listed.

To obtain a solution of the system of equations (12) with the boundary conditions listed in Table 1, a related initial value problem may be expressed as

$$d\bar{Z}/d\xi = \bar{H}(\xi, \bar{Z}; \lambda, \alpha, \gamma) \tag{13a}$$

TABLE 1

Coefficient matrices M and N of boundary conditions

Type of	M	N
Hinged immovable	$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -(R/2)\gamma\lambda & 0 & 1/R & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu/R & 1 \end{vmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & 1 \end{pmatrix}$
Hinged movable	$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -(R/2)\gamma\lambda & 0 & 1/R & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu/R & 1 \end{vmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

where

$$Z(\xi)|_{\xi=R} = [z_1, z_2, z_3, z_4, z_5, z_6]^{\mathrm{T}}_{\xi=R}$$
$$= [1, 0, \eta_1, -(\eta_1/R) + (R/2)\gamma\lambda, \eta_2, (\nu/R)\eta_2]^{\mathrm{T}}. \quad (13b)$$

Equation (13b) represents the initial value vector constructed from the boundary and normalization conditions at $\xi = R$, and η_1, η_2 and λ are unknown initial-value parameters.

A solution to the initial value problem (13) may be symbolically written as

$$\bar{Z}(\xi) = \bar{Z}(R) + \int_R^{\xi} \bar{H}(\xi, \bar{Z}; \bar{\eta}, \alpha, \gamma)\, \mathrm{d}\xi \quad (14)$$

where $\bar{\eta} = [\eta_1\ \eta_2\ \lambda]^{\mathrm{T}}$ is the unknown vector related to the missing initial values and the frequency. Given the parameters α and γ, the components of $\bar{\eta}$ are searched for such that a solution to equations (14) also satisfies the boundary conditions (12c): i.e.,

$$N\bar{Z}(1; \bar{\eta}, \alpha, \gamma) = 0. \quad (15)$$

Also in the other manuscript the method shotting was used.

For example;

Noting $\delta\tilde{W}$ and $\delta\tilde{U}$ can be arbitrary, one obtains the following nonlinear ordinary differential equations

$$\omega^2\eta^2\tilde{U}+9\left(\Delta\tilde{U}-\frac{1}{R^2}\tilde{U}+\tilde{W}_{,R}\tilde{W}_{,RR}+\frac{1-\nu}{2R}\tilde{W}_{,R}^2\right)=0 \tag{33}$$

$$(1+\kappa)\Delta\Delta\tilde{W}-\omega^2\left(\tilde{W}-\frac{1}{12}\eta^2\Delta\tilde{W}\right)-9\frac{1}{R}[R\tilde{W}_{,R}L(\tilde{W},\tilde{U})]_{,R}=0 \tag{34}$$

Similarly, the associated boundary conditions Eqs. (28) and (29) are transformed to the following:

$$\tilde{W}(0)=\beta,\ \tilde{W}_{,R}(0)=0,\ \tilde{U}(0)=0,\ \lim_{R\to 0}(\Delta\tilde{W})_{,R}=0 \tag{35}$$

$$\tilde{W}(1) = 0,\ \zeta(1+\kappa)\tilde{W}_{,RR}(1) + (\nu-\kappa)\tilde{W}_{,R}(1) = 0,\ \tilde{U}(1) = 0 \quad (36)$$

Herein, β represents the dimensionless transverse amplitude of the center of the deflected plate.

The above field equations together with the boundary conditions can be summarized as seven coupled set of first order, nonlinear ordinary differential equations as follows [43]

$$\frac{d\mathbf{Y}}{dR} = \mathbf{H}(R, \mathbf{Y}; \beta, \kappa, \eta),\ R \in (\Delta R, 1) \quad (37)$$

$$\mathbf{B}_1 \mathbf{Y}(\Delta R) = \left\{\beta, 0, 0, 0\right\}^{\mathrm{T}},\ \mathbf{B}_2 \mathbf{Y}(1) = \left\{0, 0, 0\right\}^{\mathrm{T}} \quad (38)$$

with the denotations of the forms

$$\mathbf{Y} = \left\{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7\right\}^{\mathrm{T}} = \left\{\tilde{W}, \tilde{W}_{,R}, \tilde{W}_{,RR}, \tilde{W}_{,RRR}, \tilde{U}, \tilde{U}_{,R}, \omega^2\right\}^{\mathrm{T}}$$

$$\mathbf{H} = \left\{Y_2, Y_3, Y_4, \varphi_1, Y_6, \varphi_2, 0\right\}^{\mathrm{T}}$$

$$\begin{aligned}\varphi_1 = {} & \frac{9}{1+\kappa}\left[\frac{1}{R}Y_2\left(\frac{1}{R}Y_5 + \nu Y_6 + \frac{1}{2}\nu Y_2^2\right) + Y_3\left(Y_6 + \frac{\nu}{R}Y_5 + \frac{1}{2}Y_2^2\right)\right] \\ & + \frac{1}{1+\kappa}Y_7\left[Y_1 - \eta^2\left(Y_2 Y_5 + \frac{1}{12}Y_3 + \frac{1}{12R}Y_2\right)\right] \\ & - \frac{2}{R}Y_4 + \frac{1}{R^2}Y_3 - \frac{1}{R^3}Y_2\end{aligned}$$

$$\varphi_2 = -\frac{1}{R}Y_6 + \frac{1}{R^2}Y_5 - Y_2 Y_3 - \frac{1-\nu}{2R}Y_2^2 - \frac{1}{9}\eta^2 Y_5 Y_7$$

Here, the parameter ΔR is a very small positive quantity introduced to avoid a singularity when R shrinks to zero in numerical computation. $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are matrixes of order 4×7 and 3×7, relating to the two individual boundaries, with all their components vanish except for

$$B_1(1,1) = B_1(2,2) = B_1(3,4) = B_1(4,5) = 1,\ B_1(3,3) = \frac{1}{\Delta R}$$

$$B_2(1,1) = B_2(3,5) = 1,\ B_2(2,2) = \nu - \kappa,\ B_2(2,3) = \zeta(1+\kappa)$$

The Eqs. (37) and (38) come down to a nonlinearly spatial two-point boundary problem, which can be solved by the shooting method [43], with the frequency parameter ω being an unknown constant function and the central maximum amplitude β being a control parameter. Consequently, for prescribed parameters β, κ and η, the characteristic relation of frequency versus amplitude is obtained as

$$\omega^2 = Y_7(\beta, \kappa, \eta) \quad (39)$$

The natural frequency $\omega = \omega_0$ can be achieved by setting the amplitude parameter β to be a very small value, and the β-dependent family of solutions for Eqs. (37) and (38) can be obtained by the analytical continuation method [36,37], if β is repeatedly increased by a small step Using the shooting method, the higher modes can be captured by changing the starting guess values during the iterative procedure.

Also in the other manuscript INITIAL VALUE METHOD was used

For example;.

$$cf'' + c\left(1 - \frac{\eta'}{\eta}\xi\right)\frac{1}{\xi}f' - \left(1 - \frac{\eta'}{\eta}\xi\, v\right)\frac{1}{\xi^2}f + \frac{\eta}{2\xi}(g')^2 = 0, \quad (5)$$

$$A_1 g'''' + A_2 g''' + A_3 g'' + A_4 g - \eta\lambda\frac{(c - v^2)}{(1 - v^2)}g - \frac{9(c - v^2)}{\xi}\alpha(g'f)' = \frac{(c - v^2)}{\sqrt{a}}Q^*, \quad (8)$$

where

$$A_1 = c\eta^3, \quad A_2 = \frac{2c}{\xi}\eta^3 + 6c\eta'\eta^2, \quad A_3 = -\frac{1}{\xi^2}\eta^3 + \frac{3(2c + v)}{\xi}\eta'\eta^2 + 3c\eta''\eta^2 + 6cnn'^2,$$

$$A_4 = \frac{1}{\xi}\eta^3 - \frac{3\eta'\eta^2}{\xi^2} + [3\eta''\eta^2 + 6\eta(\eta')^2]\frac{v}{\xi}, \quad \alpha = \left(A\frac{a}{h_0}\right)^2, \quad \lambda = \omega^2, \quad A^* = \frac{Aa}{h_0}p.$$

TABLE 1

Boundary conditions

Type of edge	Boundary condition at edge $\xi_1 = R$ or 1	
Clamped Immovable	$g = 0$ $g' = 0$	$cf' - (v/\xi)f = 0$
Clamped Movable	$g = 0$ $g' = 0$	$(f/\xi = 0)$
Hinged Immovable	$g = 0$ $cg'' + (v/\xi)g' = 0$	$cf' - (v/\xi)f = 0$
Hinged Movable	$g = 0$ $cg'' + (v/\xi)g' = 0$	$(f/\xi = 0)$
Free	$cg'' + (v/\xi)g' = 0$ $cg''' + c[1/\xi + 3\eta'/\eta]g''$ $-[1/\xi 2 - (3\eta'/\eta)v/\xi]g' = 0$	$(f/\xi) = 0$

3.2. INITIAL VALUE METHOD

The field equations (5) and (8) can be written as a system of six first order non-linear differential equations,

$$\mathrm{d}\bar{Y}/\mathrm{d}\xi = \bar{H}(\xi, \bar{Y}, \alpha, \lambda, Q^*), \qquad R < \xi < 1, \tag{9}$$

where

$$\bar{Y}(\xi) = \begin{Bmatrix} g \\ g' \\ g'' \\ g''' \\ f \\ f' \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} .$$

and $\bar{H}$ is the appropriately defined (6×1) vector function

$$\bar{H} = \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ -\dfrac{A_2}{A_1} y_4 - \dfrac{A_3}{A_1} y_3 - \dfrac{A_4}{A_1} y_2 + \dfrac{\eta\lambda}{A_1} \dfrac{(c - \nu^2)}{(1 - \nu^2)} y_1 \\ + \dfrac{9(c - \nu^2)}{\xi A_1} \alpha(y_3 y_5 + y_2 y_6) + \dfrac{(c - \nu^2)}{\sqrt{\alpha}\, A_1} Q^* \\ y_6 \\ \left(1 - \dfrac{\eta'}{\eta} \nu\xi\right) \dfrac{y_5}{c\xi^2} - \left(1 - \dfrac{\eta'}{\eta}\xi\right) \dfrac{y_6}{\xi} - \dfrac{\eta}{2\xi c}(y_2)^2 \end{Bmatrix}.$$

The boundary conditions for the example problems considered are
(1) $\xi = R$ (clamped immovable edge):

$$y_1 = 0, \qquad y_2 = 0, \qquad c y_6 - (\nu/R) y_5 = 0; \tag{10}$$

(2) $\xi = 1$ (free edge):

$$c y_3 + \nu y_2 = 0, \qquad y_5 = 0, \qquad c y_4 + c[1 + 3\eta'/n] y_3 - (1 - 3\eta'\nu/\eta) y_2 = 0. \tag{11}$$

Also, a unique relationship between α and λ is assured by introducing a normalization condition

$$y_1(\xi = 1) = 1. \tag{12}$$

Thus, equations and boundary conditions (9), (10), (11) and (12) can be summarized as follows:

$$\mathrm{d}\bar{Y}/\mathrm{d}\xi = \bar{H}(\xi; \alpha, \lambda, Q^*), \quad R < \xi < 1, \tag{13}$$

$$\mathbf{M}\bar{Y}(1) = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \xi = 1, \tag{14}$$

$$\mathbf{N}\bar{Y}(R) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \xi = R, \tag{15}$$

where

$$\mathbf{M} = \begin{Bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & c & 0 & 0 & 0 \\ 0 & -\left(1 - \dfrac{3\eta'}{\eta}\nu\right) & c\left(1 + \dfrac{3\eta'}{\eta}\right) & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{Bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{\nu}{R} & c \end{bmatrix}.$$

The system of equations, equations (13), (14) and (15) forms a non-linear eigenvalue problem, and is studied most conveniently by introduction of a related initial value problem:

$$\mathrm{d}\bar{Z}/\mathrm{d}\xi = \bar{H}(R < \xi < 1, \bar{Z}; \alpha, \lambda, Q^*), \tag{16}$$

$$\bar{Z}(1) = [1, \eta_1, -(\nu\eta_1)/c, (1 + \nu)\eta_1/c, 0, \eta_2]^T. \tag{17}$$

Among the components of the initial value vector of equation (17), η_1 and η_2 are missing conditions. Thus a solution of the initial value problem, equations (16) and (17), is symbolically indicated by

$$\bar{Z}(\xi) = \bar{Z}(1) + \int_1^\xi \bar{H}(\zeta; \bar{Z}; \eta_1, \eta_2, \lambda; \alpha, Q^*)\,\mathrm{d}\zeta. \tag{18}$$

Now, for known parameters, α^0 and Q_0^*, one seeks values of the missing parameters η_1, η_2 and λ, such that the corresponding solution of equations (16) satisfies the three final conditions at $\xi = R$, equation (15):

$$\mathbf{N}\bar{Z}(\xi = R; \eta_1, \eta_2, \lambda; \alpha^0, Q_0^*) = \bar{0}. \tag{19}$$

It is now apparent that solving the eigenvalue problem, equations (5) and (8), is equivalent to finding a root of the vector equation, equation (19). The solution of equation (19) for the unknown vector, $\bar{s} = (\eta_1, \eta_2, \lambda)$, can be accomplished by a direct application of a Newton method [9]. Starting from an estimated initial vector $\bar{s}^{(0)}$, and the given parameters α^0 and Q_0^*, the convergent sequence

$$\bar{s}^{(k+1)} = \bar{s}^{(k)} + \Delta\bar{s}^{(k)}, \quad k = 0, 1, 2, \ldots, \tag{20}$$

is generated, provided the following linear corrector exists:

$$\Delta\bar{s}^{(k)} = -[(N)\mathbf{T}^{(k)}]^{-1}[(N)\bar{Z}(R; \bar{s}^{(k)}; \alpha^0, Q_0^*)]. \tag{21}$$

The Fréchet derivative, $\mathbf{T}^{(k)}$, is defined as $\mathbf{T}^{(k)} = (\partial\bar{Z}/\partial\bar{s})^{(k)}_{\xi=0}$.

The analysis of the non-linear free oscillation problem is completed when the functional $\bar{s} = \bar{s}(\alpha; Q^* = 0)$ is established. The non-linear forced oscillation problem is also completed when the functional $\bar{s} = \bar{s}(\alpha, Q^*)$ is found. These functionals can be obtained by utilization of the method of continuation. The conditions sufficient to guarantee the existence, continuity and uniqueness of the solution, $\bar{s}(\alpha, Q^*)$, of equation (19) have been given by Ficken [14].

When time, τ, is equal to $\pi/2\omega$, a maximum excursion occurs and the expressions for the stresses are as follows:

$$\sigma_r^b = -\frac{6M_r}{h^2} = \left(\frac{h_0}{a}\right)^2 \frac{\eta(\xi)}{2a_{22}(c-\nu^2)}\left[cg'' + \frac{\nu}{\xi}g'\right]\sqrt{\alpha},$$

$$\sigma_\theta^b = -\frac{6M_\theta}{h^2} = \left(\frac{h_0}{a}\right)^2 \frac{\eta(\xi)}{2a_{22}(c-\nu^2)}\left[\frac{1}{\xi}g' + \nu g''\right]\sqrt{\alpha},$$

$$\sigma_r^m = \frac{N_r}{h} = \frac{(\alpha/a^2)}{a_{22}\eta(\xi)}\frac{f}{\xi}, \quad \sigma_\theta^m = \frac{N_\theta}{h} = \frac{(\alpha/a^2)}{a_{22}\eta(\xi)}f'. \tag{22}$$

Also in the other manuscript **Perturbation** was used For example;.

$$L_\lambda(\bar{w}) - \frac{3}{4}\frac{d}{dx}\left(\bar{v}\frac{d\bar{w}}{dx}\right) - \omega^2\bar{w} = 0 \tag{2.12}$$

$$G(\bar{v}) = -(d\bar{w}/dx)^2/2 \tag{2.13}$$

$$\frac{d\bar{w}}{dx} = 0, \frac{d^3\bar{w}}{dx^3} + \frac{1}{x}\frac{d^2\bar{w}}{dx^2} - \frac{m\omega^2}{2}x\bar{w} = 0, \qquad \text{at } x = c \tag{2.14a,b}$$

$$\bar{v} - \frac{x}{\mu}\frac{d\bar{v}}{dx} = 0, \qquad \text{at } x = c \tag{2.14c}$$

$$\bar{w} = 0, \quad \frac{d\bar{w}}{dx} = 0, \quad \bar{v} - \frac{x}{\mu}\frac{d\bar{v}}{dx} = 0, \qquad \text{at } x = 1 \tag{2.15a,b,c}$$

In which the different operators L_λ and G are defined as

$$L_\lambda = \frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} + \frac{\lambda}{1-\mu}\right) \tag{2.16}$$

$$G = x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}x \tag{2.17}$$

III. Perturbation Solution of the Problem

Because there are nonlinear and coupled terms in the governing equations, it is not easy to obtain the exact solution of the problem. Usually, a "shooting" method[4] or a perturbation method[5] can be used to get an approximate solution. By the former method a boudary value problem can be translated into an initial problem and by the latter nonlinear equations can be linearized into a series of linear ones. Now, we use the latter method to get the solution of equations (2.12) – (2.15). First, we take a perturbation parameter as

$$\varepsilon=\bar{w}(c)\equiv\sqrt{12(1-\mu^2)}\,\bar{W}(a)/h \tag{3.1}$$

In which $\bar{W}(a)$ is the initial deflection (or amplitude) of the inner edge of the plate. Then the related unknown quantities can be expressed in the form of power series in parameter ε. (For convenience, we will omit the bar above the letters.)

$$w=\varepsilon w_1+\varepsilon^3 w_3+\varepsilon^5 w_5+\varepsilon^7 w_7+\cdots \tag{3.2}$$

$$v=\varepsilon^2 v_2+\varepsilon^4 v_4+\varepsilon^6 v_6+\varepsilon^8 v_8+\cdots \tag{3.3}$$

$$\omega^2=\omega_0^2+k_2\varepsilon^2+k_4\varepsilon^4+k_6\varepsilon^6+\cdots \tag{3.4}$$

Inserting (3.2) (3.3) (3.4) into (2.12) – (2.15), using (3.1) and comparing the coefficients of the same power, it is easy to obtain the following linear boundary value problems

$$\varepsilon:\ \left.\begin{array}{l} L_\lambda(w_1)-\omega_0^2 w_1=0;\ w_1'(c)=0,\ w_1(c)=1 \\ w_1'''(c)+\frac{1}{c}w_1''(c)-\frac{c}{2}m\omega_0^2 w_1(c)=0;\ w_1(1)=0,\ w_1'(1)=0 \end{array}\right\} \tag{3.5}$$

$$\varepsilon^2:\ G(v_2)=-\frac{1}{2}\left(\frac{dw_1}{dx}\right)^2;\ v_2(c)-\frac{c}{\mu}v_2'(c)=0,\ v_2(1)-\frac{1}{\mu}v_2'(1)=0 \tag{3.6}$$

$$\varepsilon^3:\ \left.\begin{array}{l} L_\lambda(w_3)-\omega_0^2 w_3-\frac{3}{4}\frac{1}{x}\frac{d}{dx}\left(v_2\frac{dw_1}{dx}\right)-k_2 w_1=0 \\ w_3(c)=0,\ w_3'(c)=0 \\ w_3'''(c)+\frac{1}{c}w_3''(c)-\frac{c}{2}m(\omega_0^2 w_3(c)+k_2 w_1(c))=0 \\ w_3(1)=0,\ w_3'(1)=0 \end{array}\right\} \tag{3.7}$$

$$\varepsilon^4:\ G(v_4)=-\frac{dw_1}{dx}\frac{dw_3}{dx};\ v_4(c)-\frac{c}{\mu}v_4'(c)=0,\ v_4(1)-\frac{1}{\mu}v_4'(1)=0 \tag{3.8}$$

$$\varepsilon^5:\ \left.\begin{array}{l} L_\lambda(w_5)-\omega_0^2 w_5-\frac{3}{4}\frac{1}{x}\frac{d}{dx}\left(v_2\frac{dw_3}{dx}+v_4\frac{dw_1}{dx}\right)-k_2 w_3-k_4 w_1=0 \\ w_5(c)=0,\ w_5'(c)=0 \\ w_5'''(c)+\frac{1}{c}w_5''(c)-\frac{c}{2}m(\omega_0^2 w_5(c)+k_2 w_3(c)+k_4 w_1(c))=0 \\ w_5(1)=0,\ w_5'(1)=0 \\ \cdots \end{array}\right\} \tag{3.9}$$

where the prime denotes the differential with respect to x. Through solving the above linear boundary value problems, the perturbation solution of (2.12) – (2.15) with the form of (3.2) – (3.4) can be obtained.