Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	1	-3	19
x_1	1	0	0	2	3
x_2	0	1	2	0	4

Table 12: Simplex tableau for the problem.

Generate the simplex tableau associated with the basis $B = (\underline{P}_1, \underline{P}_2)$.
$\underline{x}_B = (x_1, x_2)^T$ and $\underline{c}_B = (1, 4)^T$.
Also $B = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}$.
Thus $\underline{x}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1}\underline{b} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Note we are lucky here since the variables are positive meaning the initial solution is feasible.
$B^{-1}A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$
$\underline{c}_B^T B^{-1} A - \underline{c}^T = (0, 0, 1, -3)$. The objective function is $z = \underline{c}_B^T B^{-1}\underline{b} = c_B^T \underline{x}_B = 19)$. This gives the simplex tableau shown in the table.

2.4 The Revised Simplex Method in Matrix Form

The matrix representation is more accurate than the use of elementary row and column operations since at each step the tableau is generated from the original data and the B^{-1} matrix, whereas in the first formulation the elementary row and column operations were applied sequentially which could lead to a build up in round off error.
Optimality Condition
From the Simplex tableau in matrix form, an increase in the non-basic variable x_j will increase (improve for maximisation) the value of z only if $\underline{c}_B^T B^{-1} \underline{P}_j - c_j$ is strictly negative. The coefficient needs to be strictly positive for minimisation. The simplex method uses the rule of thumb to select the most negative (positive) coefficient in the case of maximisation (minimisation). The *entering* variable is that corresponding to that coefficient.

Feasibility Condition
Once the entering vector P_j has been chosen, the feasibility condition works

in the same way as for the tableau method. In matrix form this can be stated as :
The *leaving* variable x_k is that variable corresponding to the index k (from all the possible i values) that minimises $\frac{(B^{-1}\underline{b})_i}{(B^{-1}\underline{P}_j)_i}$ subject to $(B^{-1}\underline{P}_j)_i > 0$.

The algorithm proceeds by construction of a new basic variables matrix B at each step by removing column $\underline{P}_k$ from B and variable x_k from $\underline{x}_B$ and replacing the column $\underline{P}_k$ in B with $\underline{P}_j$ and variable x_k from $\underline{x}_B$ with x_j. The new basic variables matrix is B_{new} and the new solution for the basic variables is $\underline{x}_{Bnew} = B_{new}^{-1}\underline{b}$.

Example: The Ready Mix problem revisited
The problem can be written in matrix form as

Maximise $z = (5,4,0,0,0,0)(x_1,x_2,x_3,x_4,x_5,x_6)^T$ subject to

$$\begin{pmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix}$$

Iteration 0
$\underline{x}_B = (x_3,x_4,x_5,x_6)^T$, $\underline{c}_B = (0,0,0,0)^T$. The matrix B is the identity so that $B^{-1} = I$.
Thus $\underline{x}_B = (24,6,1,2)^T$ and $z = \underline{c}_B^T\underline{x}_B = 0$.

Optimality Computation
$\underline{c}_B^T B^{-1} = (0,0,0,0)$ and $\{z_j - c_j\}_{j=1,2} = \underline{c}_B^T B^{-1}(\underline{P}_1, \underline{P}_2) - (c_1, c_2) = (-5,-4)$.
The most negative is -5 so $\underline{P}_1$ is the entering vector.

Feasibility Computation
$\underline{x}_B = (x_3,x_4,x_5,x_6)^T = (24,6,1,2)^T$ and $B^{-1}\underline{P}_1 = (6,1,-1,0)^T$
hence, $min\{\frac{24}{6}, \frac{6}{1}, \frac{1}{-1}, \frac{2}{0}\} = 4$, corresponding to x_3 so $\underline{P}_3$ becomes the leaving vector.
Iteration 1
$\underline{x}_B = (x_1,x_4,x_5,x_6)^T$, $\underline{c}_B = (5,0,0,0)^T$. The matrix $B = (\underline{P}_1, \underline{P}_4, \underline{P}_5, \underline{P}_6) =$

$\begin{pmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The inverse is given by

$$B^{-1} = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & 1 & 0 & 0 \\ \frac{1}{6} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $\underline{x}_B = B^{-1}\underline{b} = (4, 2, 5, 2)^T$ and $z = \underline{c}_B^T \underline{x}_B = 20$.

Optimality Computation
$\underline{c}_B^T B^{-1} = (\frac{5}{6}, 0, 0, 0)$ and $\{z_j - c_j\}_{j=2,3} = c_B^T B^{-1}(\underline{P}_2, \underline{P}_3) - (c_2, c_3) = (-\frac{2}{3}, \frac{5}{6})$. The most negative is -2/3 so $\underline{P}_2$ is the entering vector.

Feasibility Computation
$\underline{x}_B = (x_1, x_4, x_5, x_6)^T = (4, 2, 5, 2)^T$ and $B^{-1}\underline{P}_2 = (\frac{2}{3}, \frac{4}{3}, \frac{5}{3}, 1)^T$
hence, $min\{\frac{4}{\frac{2}{3}}, \frac{2}{\frac{4}{3}}, \frac{5}{\frac{5}{3}}, \frac{2}{1}\} = \frac{3}{2}$, corresponding to x_4 so $\underline{P}_4$ becomes the leaving vector.

Iteration 2
$\underline{x}_B = (x_1, x_2, x_5, x_6)^T$, $\underline{c}_B = (5, 4, 0, 0)^T$. The matrix $B = (\underline{P}_1, \underline{P}_2, \underline{P}_5, \underline{P}_6)$.

$$B^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{3}{4} & 0 & 0 \\ \frac{3}{8} & -\frac{5}{4} & 1 & 0 \\ \frac{1}{8} & -\frac{3}{4} & 0 & 1 \end{pmatrix}$$

Thus $\underline{x}_B = B^{-1}\underline{b} = (3, \frac{3}{2}, \frac{5}{2}, \frac{1}{2})^T$ and $z = \underline{c}_B^T \underline{x}_B = 21$.

Optimality Computation
$\underline{c}_B^T B^{-1} = (\frac{3}{4}, \frac{1}{2}, 0, 0)$ and $\{z_j - c_j\}_{j=3,4} = c_B^T B^{-1}(\underline{P}_3, \underline{P}_4) - (c_3, c_4) = (\frac{3}{4}, \frac{1}{2})$. There are no negative numbers so the $\underline{x}_B$ is optimal and the computations end, giving $x_1 = 3,\ x_2 = \frac{3}{2},\ z = 21$.

Note that the final inverse matrix B^{-1} has the same entries as in the simplex tableau, shown in table 3.

2.5 Duality and Sensitivity Analysis

The concept of duality is best seen through an example. Remember the dietician's problem.