NOTES ON THE STABILIZATION OF AN INVERTED PENDULUM

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Abstract. These notes present my take on a small part of the analysis of the stability of a pendulum in Butikov [2]. I have also been influenced by the more general analysis in Landau and Lifshitz [4], section 30 (page 93) and the problems at the end of that section. The review in Section 1 is based on Hale [3].

We wish to analyze the effect on a pendulum's stability of applying high frequency and small amplitude displacements to the pivot.

We limit the analysis to motion in a vertical plane (i.e., two-dimensional motion) of an ordinary pendulum consisting of a mass m attached to a massless bar of length l.

Introduce the Cartesian coordinates (x, y), where x is horizontal and y points *downward*. The origin is the nominal location of the pivot. Let ϕ be the angle, measured counterclockwise, that the pendulum makes relative to the y axis.

1. A review of stability

Given a function $f : \mathbf{R} \to \mathbf{R}$, consider the following differential equation for $x(t)$:

$$\ddot{x} = f(x),$$

Suppose f has a potential, U; that is, $f(x) = -\frac{d}{dx}U(x)$. Then the equilibria of the differential equation are the critical points of U. If a critical point is a local minimum, then the corresponding equilibrium is stable. If it is a local maximum, then the corresponding equilibrium is unstable. This is formally proved in Hale, page 172. The crux of the argument is the observation that the equation implies that

$$\dot{x}\ddot{x} = -\frac{d}{dx}U(x)\dot{x},$$

whence

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + U(x)\right) = 0,$$

and consequently,

$$\frac{1}{2}\dot{x}^2 + U(x) = \text{constant along the orbits.}$$

The level curves of $\frac{1}{2}\dot{x}^2 + U(x)$ look like ellipses near the minima of U, and they look like saddles near the maxima of U.

As a specific example consider the equation of a simple pendulum of length l:

$$ml\ddot{\phi} + mg\sin\phi = 0.$$

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We note that this is equivalent to:

$$ml\ddot{\phi} = -\frac{d}{d\phi}U(\phi),$$

where

$$U(\phi) = mg(1 - \cos\phi)).$$

U has a local minimum at $\phi = 0$ and a local maximum at $\phi = \pi$. It follows that the hanging down equilibrium is stable and the upright equilibrium is unstable.

2. Vertical oscillation of the support

Suppose that the pivot oscillates vertically according to

$$y = a\sin\omega t. \tag{1}$$

Then it is not hard to show that the pendulum's equation of motion is

$$\ddot{\phi} + \frac{g}{l}\sin\phi + \frac{a\omega^2}{l}\sin\phi\sin\omega t = 0. \tag{2}$$

The fast oscillations of the pivot result in a motion $\phi(t)$ which is a superposition of a slow motion, $\psi(t)$, and a fast, small amplitude oscillations $\delta(t)$ of frequency ω. Thus, we seek a solution to (2) of the form

$$\phi(t) = \psi(t) + \delta(t).$$

The oscillations $\delta(t)$ can be characterized by following the idea expounded in [2] and illustrated in Figure 1 which depicts the pendulum in when its arm oscillates rapidly about the nominal angle ψ with the vertical. Applying the *law of sines* within the triangle shown, we get

$$\frac{\sin\delta}{a\sin\omega t} = \frac{\sin\psi}{l}.$$

Assuming that a/l is small, then δ is small, so that $\sin\delta \approx \delta$, this leads to

$$\delta = \frac{a}{l}\sin\psi\sin\omega t \qquad \text{(assuming } a/l \ll 1). \tag{3}$$

We are going to need δ's second derivative soon, so let's calculate it right now. We have:

$$\dot{\delta} = \frac{a}{l}\big[\dot{\psi}\cos\psi\sin\omega t + \omega\sin\psi\cos\omega t\big],$$
$$\ddot{\delta} = \frac{a}{l}\big[\ddot{\psi}\cos\psi\sin\omega t - \dot{\psi}^2\sin\psi\sin\omega t + 2\omega\dot{\psi}\cos\psi\cos\omega t - \omega^2\sin\psi\sin\omega t\big].$$

We are interested in high frequency oscillations of the pivot, that is, $\omega \gg 1$. Therefore, the term with ω^2 in the expression above dominates the rest. We conclude that

$$\ddot{\delta} \approx -\frac{a\omega^2}{l}\sin\psi\sin\omega t \qquad \text{(assuming } \omega \gg 1).$$

Now we go to the differential equation (2) and replace ϕ by $\psi + \delta$, and replace $\sin\phi$ with its Taylor series approximation

$$\sin\phi = \sin(\psi + \delta) \approx \sin\psi + \delta\cos\psi.$$

We get:

$$\ddot{\psi} + \ddot{\delta} + \frac{g}{l}\big[\sin\psi + \delta\cos\psi\big] + \frac{a\omega^2}{l}\big[\sin\psi + \delta\cos\psi\big]\sin\omega t.$$

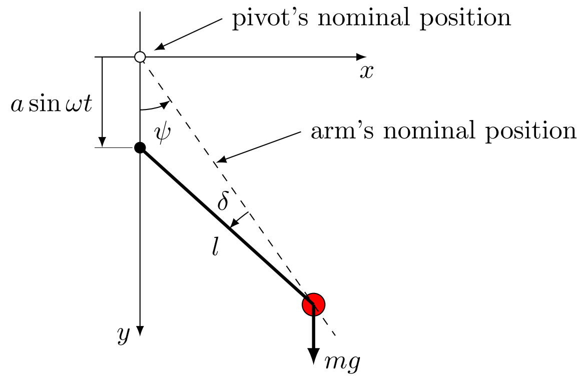

FIGURE 1. The pivot oscillates vertically according to $a \sin \omega t$ about the pendulum's nominal pivot. When the pendulum's arm makes a "nominal" angle ψ with the vertical, the angle actually oscillates rapidly in the range $\psi \pm \delta$ as shown. Beware that the pivot's displacement is exaggerated; we assume that a/l is very small in our computations.

We multiply out everything and replace $\ddot{\delta}$ with the expression obtained above, and arrive at

$$\ddot{\psi} - \frac{a\omega^2}{l} \sin\psi \sin\omega t + \frac{g}{l} \sin\psi + \frac{g}{l} \delta \cos\psi + \frac{a\omega^2}{l} \sin\psi \sin\omega t + \frac{a\omega^2}{l} \delta \cos\psi \sin\omega t = 0.$$

The second and fifth terms cancel, leaving us with

$$\ddot{\psi} + \frac{g}{l} \sin\psi + \frac{g}{l} \delta \cos\psi + \frac{a\omega^2}{l} \delta \cos\psi \sin\omega t = 0.$$

In the last term we substitute for δ from (3):

$$\ddot{\psi} + \frac{g}{l} \sin\psi + \frac{g}{l} \delta \cos\psi + \frac{a^2\omega^2}{l^2} \sin\psi \cos\psi \sin^2\omega t = 0.$$

Now, we average the equation over one fast period, that is $2\pi/\omega$, during which time the slow function ψ effectively remains unchanged. Thus, averaging does not affect the first and second terms. The third term goes away since, in view of (3), δ is proportional to $\sin\omega t$, whose average over one period is zero. Averaging the last term requires calculating the integral

$$\frac{1}{2\pi/\omega} \int_0^{2\pi/\omega} \sin^2\omega t \, dt = \frac{2\pi}{\omega} \int_0^{2\pi/\omega} \frac{1}{2}(1 - \cos 2\omega t) \, dt = \frac{1}{2}.$$

We conclude that

$$\ddot{\psi} + \frac{g}{l} \sin\psi + \frac{a^2\omega^2}{2l^2} \sin\psi \cos\psi = 0,$$

or equivalently

$$\ddot{\psi} + \frac{g}{l} \left[1 + \frac{a^2\omega^2}{2lg} \cos\psi\right] \sin\psi = 0.$$

To make sense of the combination of the coefficients that appear here. let us introduce the notation

$$\omega_0 = \sqrt{\frac{g}{l}}. \tag{4}$$

This is the angular frequency of the pendulum's small oscillations about the handing down positions. Putting the previous equation in terms of ω_0 we get

$$\ddot{\psi} + \omega_0^2 \left[1 + \frac{1}{2} \left(\frac{a}{l} \cdot \frac{\omega}{\omega_0} \right)^2 \cos\psi \right] \sin\psi = 0. \tag{5}$$

3. Conclusions

When the pendulum's bob is lower that its pivot, that is, $\phi < \pi/2$, the expression in the square bracket in (5) is greater than 1, which tells us that the effect of fast oscillations of the base is tantamount to increasing the force of gravity. On the other hand, when the bob is above the pivot, that is $\phi > \pi/2$, the expression in the square bracket is less than one, which is tantamount to decreasing the force of gravity. In fact, if

$$\frac{1}{2} \left(\frac{a}{l} \cdot \frac{\omega}{\omega_0} \right)^2 > 1,$$

then the expression in the square bracket may even become negative, which amount to reversing the direction of the gravitational pull! When that happens, the pendulum will find a stable equilibrium in the upright position. Perhaps a better way of writing the stability condition above is

$$\frac{a}{l} \cdot \frac{\omega}{\omega_0} > \sqrt{2}. \tag{6}$$

Note that the two terms in the multiplication on the left are (a) the dimensionless amplitude of the pivots motion, and (b) the dimensionless angular frequency of the pivot.

4. A double pendulum

This section is incomplete

Acheson [1] establishes a stability criterion for the inverted stability of N pendulums. I will translate his notation to ours.

Consider a chain of N pendulums. Let $\omega_{\min}$ and $\omega_{\max}$ be the smallest and largest natural frequencies of small oscillations of that compound pendulum about it stable (that is, hanging down) position. Now, invert the pendulum and subject the pivot to vertical displacements as in (1). It is shown that the inverted position will be stable if

$$\frac{\sqrt{2}g}{\omega\omega_{\min}} < a < \frac{0.450g}{\omega_{\max}^2} \qquad (\text{assuming } \omega \gg \omega_{\max}). \tag{7}$$

4.1. **The case $N = 1$.**

In the case of a simple pendulum, i.e., $N = 1$, we have $\omega_{\min} = \omega_{\max} = \sqrt{g/l} = \omega_0$. Therefore (7) takes the form

$$\frac{\sqrt{2}g}{\omega\sqrt{g/l}} < a < \frac{0.450g}{g/l},$$

which simplifies to

$$\sqrt{2} < \frac{a}{l} \cdot \frac{\omega}{\omega_0} < 0.450\frac{\omega}{\omega_0},$$

where ω_0 is defined in (4). This is a tightened version of the stability criterion (6).

4.2. **The case $N = 2$.** Consider a double pendulum consisting of masses m_1 and m_2 and equal link lengths of l each. Assume m_1 is hung from the pivot and m_2 hangs from m_1. According to [1], the natural frequencies are (I haven't checked these,)

$$\omega_{\min} = \sqrt{\frac{g/l}{1+\sqrt{m}}}, \qquad \omega_{\max} = \sqrt{\frac{g/l}{1-\sqrt{m}}},$$

where $m = m_2/(m_1+m_2)$. Then the general stability criterion reduces to

$$\sqrt{2}\sqrt{1+\sqrt{m}} < \frac{a}{l}\cdot\frac{\omega}{\omega_0} < 0.450\sqrt{1-\sqrt{m}}\frac{\omega}{\omega_0},$$

where $\omega_0 = \sqrt{g/l}$, as before.

References

[1] D. J. Acheson, *A Pendulum Theorem*, Proceedings: Mathematical and Physical Sciences **443** (1993), no. 1917, 239-245.

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[4] L. D. Landau and E. M. Lifshitz, *Mechanics*: *Volume 1 of Course of Theoretical Physics*, Addison–Wesley, Reading, MA, 1960. Translated from the Russian.

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