Chaos, Solitons and Fractals 98 (2017) 189–198

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

On Hopf bifurcation in fractional dynamical systems

Amey S. Deshpande[a], Varsha Daftardar-Gejji[a,*], Yogita V. Sukale[a,b]

[a] Department of Mathematics, Savitribai Phule Pune University, Pune, 411007, India
[b] Department of Mathematics, College of Engineering Pune, Pune, 411005, India

ARTICLE INFO

Article history:
Received 21 November 2016
Revised 7 February 2017
Accepted 15 March 2017

Keywords:
Fractional dynamics
Caputo derivative
Hopf bifurcation
Chaos

ABSTRACT

Fractional order dynamical systems admit chaotic solutions and the chaos disappears when the fractional order is reduced below a threshold value [1]. Thus the order of the dynamical system acts as a chaos controlling parameter. Hence it is important to study the fractional order dynamical systems and chaos. Study of fractional order dynamical systems is still in its infancy and many aspects are yet to be explored.

In pursuance to this in the present paper we prove the existence of fractional Hopf bifurcation in case of fractional version of a chaotic system introduced by Bhalekar and Daftardar-Gejji [2]. We numerically explore the (A, B, α) parameter space and identify the regions in which the system is chaotic. Further we find (global) threshold value of fractional order α below which the chaos in the system disappears regardless of values of system parameters A and B.

1. Introduction

Fractional order dynamical systems are gaining popularity due to their widespread applications [3]. The study of chaotic dynamical systems of fractional order was initiated by Grigorenko and Grigorenko [1] wherein fractional ordered Lorenz system was studied. It was shown that the order of the derivative acts as a chaos controlling parameter and below a threshold value of α, chaos disappears. These simulations were done by keeping rest of the system parameters fixed. Since then there has been increasing interest in this topic and a large number of contributions have appeared in the literature which deal with fractional versions of various chaotic systems including Chen System [4], Rössler system [5], Liu system [6], financial system [7] and so on [3].

In spite of the extensive numerical work, our understanding of fractional systems is not complete and very few analytical results have been obtained. The first important result obtained regarding stability analysis of fractional systems is due to Matignon [8]. Some of the important results regarding stability of fractional systems have been summarized by Li and Zhang [9].

The system introduced by Bhalekar and Daftardar-Gejji (BG system) has been shown to be chaotic for certain values of parameters [2]. Further the forming mechanism of this system is discussed by Bhalekar [10]. The synchronization and anti-synchronization of Bhalekar – Gejji system and Liu system is done by Singh et al. [11]. The Hopf bifurcation in integer order Bhalekar – Gejji system has been explored by Aqeel and Ahmad [12]. In the present paper we prove existence of Hopf bifurcation in fractional version of BG system and explore the parameter space numerically.

The paper is organized as follows. Section 2 comprises of preliminaries and notations. Section 3 deals with fractional Hopf bifurcation and proves its existence for fractional BG system. Sections 4, 5 and Section 6 contain numerical explorations for various system parameters. Conclusions are summarized in Section 7.

2. Preliminaries

In this section, we introduce notations, definitions and preliminaries pertaining to fractional calculus and stability of fractional dynamical systems [1,9,13,14]. $N_r(a)$ denotes the neighborhood of point $a \in \mathbb{R}^n$ having radius $r > 0$. $\|.\|$ denotes standard Euclidean norm on $\mathbb{R}^n$.

Definition 1 [15]**.** The fractional integral of order $\alpha > 0$ of a real valued function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$

Definition 2 [15]**.** Caputo fractional derivative of order $\alpha > 0$ of a real valued function f is defined as

$$D^\alpha f(t) = I^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad m-1 < \alpha < m,$$
$$= f^{(m)}(t), \quad \alpha = m, \; m \in \mathbb{N}.$$

* Corresponding author.
E-mail addresses: 2009asdeshpande@gmail.com (A.S. Deshpande), vsgejji@unipune.ac.in, vsgejji@gmail.com (V. Daftardar-Gejji), yvs.maths@coep.ac.in (Y.V. Sukale).

http://dx.doi.org/10.1016/j.chaos.2017.03.034
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We assume $0 < \alpha \leq 1$ throughout the paper.

Consider the fractional order autonomous system with bifurcation parameter $\mu \in \mathbb{R}^m$

$$D^\alpha x(t) = f_\mu(x), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{1}$$

Definition 3 [9]**.** A point $e_\mu \in \mathbb{R}^n$ is called as an **equilibrium point** of (1) if $f_\mu(e_\mu) = 0$.

Definition 4 [9]**.** The system (1) is called as **locally stable** if for every $\epsilon > 0, \exists \delta > 0$ such that $x_0 \in N_\delta(e_\mu) \Rightarrow \|x_\mu(t) - x_\mu(e_\mu)\| < \epsilon$ for $t > 0$.

Definition 5 [9]**.** The system (1) is called as (locally) **asymptotically stable** if for $x_\mu(t)$ as above, $\|x_\mu(t) - x_\mu(e_\mu)\| \to 0$ as $t \to \infty$.

A system is **unstable** if it is not stable.

The linearization of (1) around the equilibrium point e_μ is given as

$$D^\alpha x(t) = Jx(t), x(0) = x_0, \tag{2}$$

where $[J]_{i,j} = \frac{\partial f_\mu^i}{\partial x_j}(e_\mu), \ 1 \leq i, j \leq n$. Clearly eigenvalues λ of J depend on the bifurcation parameter. To simplify the notation we generally drop the suffix μ for λ unless it needs to be emphasized.

Definition 6 [9]**.** The linearized system (2) is called as (locally) **linearly stable** if for each eigenvalue λ of J, $|arg(\lambda)| > \frac{\pi\alpha}{2}$.

The system (2) is said to be **linearly unstable** if $|arg(\lambda)| < \frac{\pi\alpha}{2}$, for at least one eigenvalue λ of J. The equilibrium point e_μ is defined as non-hyperbolic equilibrium point if $|arg(\lambda)| = \frac{\pi\alpha}{2}$, for some eigenvalue λ of J.

This stability criteria is due to Matignon [8], which coincides with the concept of local stability for fractional systems (1) [16].

Definition 7. Given that $\mu = \mu_0$ fixed, **threshold value** $\alpha^*_{\mu_0}$ for the system (1) is defined as

$$\alpha^*_{\mu_0} = \inf_{\alpha_c}\{\text{for } \alpha > \alpha_c, \text{system (1) is chaotic}\}. \tag{3}$$

Clearly for $\alpha < \alpha^*_{\mu_0}$ chaos in the system $D^\alpha x(t) = f_{\mu_0}(x)$ disappears.

In the next section we highlight system parameter dependence of this definition. Further we propose a definition of global threshold value for parameter μ.

3. Analysis of fractional BG system

The system is said to undergo a Hopf bifurcation when an equilibrium point switches the stability along-with creation or destruction of certain periodic orbits [13]. For the integer order system, this is known to occur when equilibrium has pair of eigenvalues that cross the imaginary axis at non-zero speed.

Due to the changed stability criteria for fractional systems [8], it is quite natural to frame the existence criteria for the fractional Hopf bifurcation as follows.

Existence criteria for Fractional Hopf Bifurcation. Consider the system of fractional differential equations given by Eq. (1) together with bifurcation parameter $\mu \in \mathbb{R}$ and $\alpha \in (0, 1]$. Let e_μ be the equilibrium point of (1) and (2) be its linearization around e_μ.

Suppose $n \times n$ matrix A has $\lambda_1(\mu)$, $\lambda_2(\mu)$,..., $\lambda_n(\mu)$ as its eigenvalues such that at least one pair of eigenvalues say $\lambda_1(\mu)$, $\lambda_2(\mu)$ is complex conjugate.

We say that (1) undergoes fractional Hopf bifurcation, if $\exists$ a critical value $\mu = \mu_h$ such that the following conditions are satisfied.

1. *$\lambda_1(\mu_h)$ and $\lambda_2(\mu_h)$ satisfy* $|\arg(\lambda_j(\mu_h))| = \frac{\pi\alpha}{2} \quad (j = 1, 2)$,
2. $|\arg(\lambda_i(\mu_h))| \neq \frac{\pi\alpha}{2}, \quad (i = 3, 4, \cdots, n)$,
3. $\frac{d}{d\mu} |\arg(\lambda_j(\mu))| \,|_{\mu=\mu_h} \neq 0, \quad (j = 1, 2)$.

First and second conditions are sometimes called as singularity conditions while the third is transversality condition.

Daftardar-Gejji and Bhalekar introduced a new dynamical system referred to as Bhalekar–Gejji (BG) system. This system exhibits chaos for certain parameter values [12]. In the present paper we investigate the fractional version of the system *i.e.*

$$D^\alpha x(t) = \omega x(t) - y^2(t), \tag{4}$$

$$D^\alpha y(t) = \mu(z(t) - y(t)), \tag{5}$$

$$D^\alpha z(t) = Ay(t) - Bz(t) + x(t)y(t), \tag{6}$$

where $0 < \alpha \leq 1$ and ω, μ, A, B are parameters. Standard values for which the system exhibits chaos are $\omega = -2.667, \ \mu = 10, \ A = 27.3, \ B = 1$. μ is generally taken as positive while ω is a negative real number.

Consider for further analysis $\omega < 0$ and $\mu > 0$. Objective of this paper is to show that Bhalekar–Gejji system satisfies the above criteria and hence the route taken by system to chaos is of fractional Hopf bifurcation. Further we extensively analyze the effect of variation of parameter in $A-B$ plane on the system. For fractional systems, fractional order α also acts as a bifurcation parameter. We study effect of variation of α over the system as well. As a result we identify the exact region in $A-B$ plane for which the system is stable and chaotic.

For $B - A > 0$, system has only one equilibrium point *i.e.* $P_1(0, 0, 0)$. At $B = A$ system undergoes **supercritical pitchfork bifurcation**, with origin turning into index 1 saddle point with formation of two new symmetrically opposite equilibrium points P_2, P_3. For $B - A \leq 0$, $P_2(B-A, \sqrt{\omega(B-A)}, \sqrt{\omega(B-A)})$ and $P_3(B-A, -\sqrt{\omega(B-A)}, -\sqrt{\omega(B-A)})$.

The Jacobian matrix of (4) is given as

$$J = \begin{pmatrix} \omega & -2y & 0 \\ 0 & -\mu & \mu \\ y & A+x & -B \end{pmatrix}. \tag{7}$$

Around origin $J|_{P_1}$ the eigenvalues are given as $\frac{1}{2}[-B-\mu \pm \sqrt{(B-\mu)^2 + 4A\mu}]$ and ω. Since $\omega < 0$, stability analysis as given above is easy to verify.

For P_2 and P_3, eigenvalues are roots of the same characteristic polynomial. Let $f(\lambda)$ denote the characteristic polynomial of P_2 and P_3. Then

$$f(\lambda) = \lambda^3 + (B + \mu - \omega)\lambda^2 - \omega(B+\mu)\lambda + 2\mu\omega(B-A) \tag{8}$$

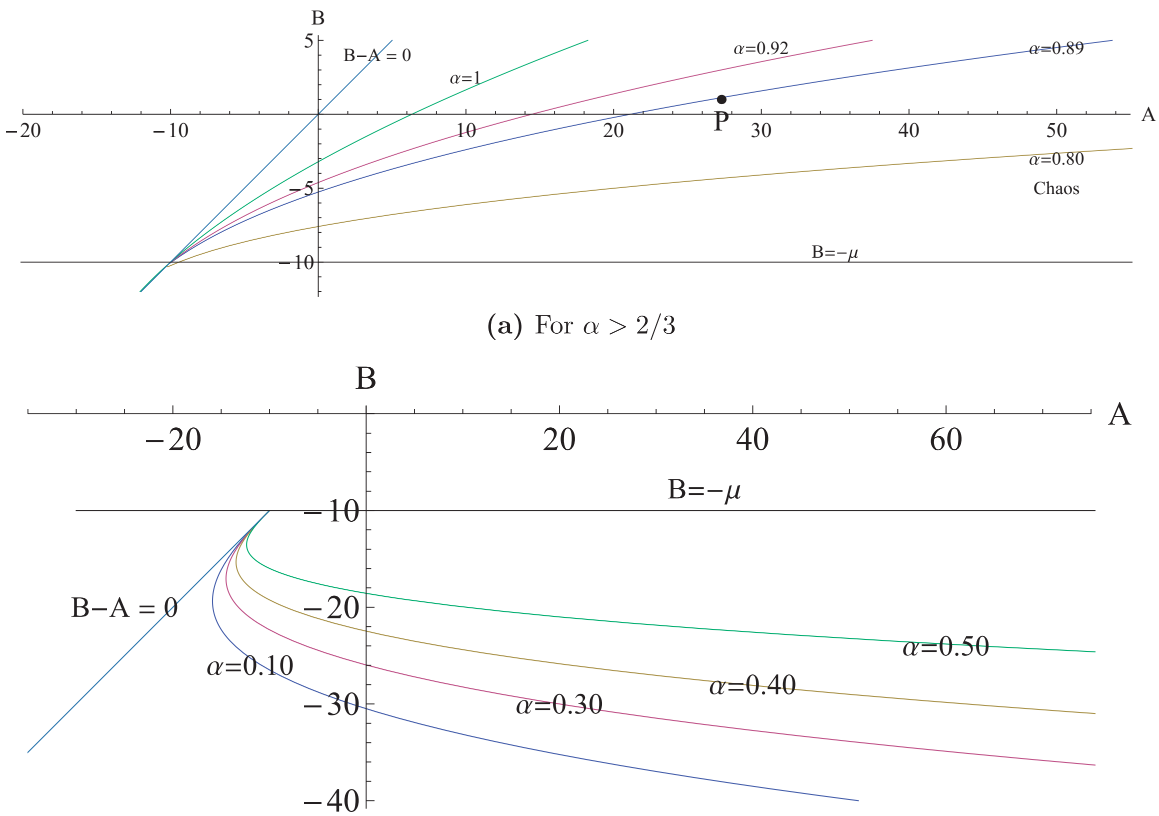
and $f(\lambda) = 0$ is the corresponding characteristic equation. Fractional Hopf bifurcation will occur when complex roots of $f(\lambda)$ will cross into the cone $|arg(\lambda)| < \frac{\pi\alpha}{2}$. Set $A = A_h$, the Hopf critical value, in this case $\lambda = re^{i\theta}$ where $\theta = \pm\frac{\pi\alpha}{2}$ will satisfy $f(\lambda) = 0$. Thus we get

$$\begin{aligned} &r^3 e^{i3\theta} + (B+\mu-\omega)r^2 e^{2i\theta} - \omega(B+\mu)re^{i\theta} + 2\mu\omega(B-A_h) = 0 \\ &r^3(\cos(3\theta) + i\sin(3\theta)) + (B+\mu-\omega)r^2(\cos(2\theta) + i\sin(2\theta)) \\ &\quad -\omega(B+\mu)r(\cos(\theta) + i\sin(\theta)) + 2\mu\omega(B-A_h) = 0. \end{aligned} \tag{9}$$

Equating real and imaginary parts in (9) we get

$$\begin{aligned} &\cos(3\theta)r^3 + (B+\mu-\omega)\cos(2\theta)r^2 \\ &\quad -\omega(B+\mu)\cos(\theta)r + 2\mu\omega(B-A_h) = 0, \end{aligned} \tag{10}$$

$$\sin(3\theta)r^3 + (B+\mu-\omega)\sin(2\theta)r^2 - \omega(B+\mu)\sin(\theta)r = 0. \tag{11}$$



(a) For $\alpha > 2/3$

(b) For $\alpha < 2/3$

Fig. 1. Curves showing Hopf critical values for $\alpha = 1,\ 0.92,\ 0.89, 0.86$ Fig. (1a) and for $\alpha = 0.10, 0.30, 0.40, 0.50$ Fig. (1b) while the line $B - A = 0$ shows the points where pitchfork bifurcation occurs. Point P is the point (27.3, 1) and line $B = -10$ forms the boundary of the chaotic region.

As we are interested in non-zero solutions, dividing (11) by r we get

$$\sin(3\theta)r^2 + (B+\mu-\omega)\sin(2\theta)r - \omega(B+\mu)\sin(\theta) = 0. \tag{12}$$

Solving (12) for r we get

$$r_{1,2} = \frac{-(B+\mu-\omega)\sin(2\theta) \pm \sqrt{\Delta}}{2\sin(3\theta)}, \tag{13}$$

where $\Delta = (B+\mu-\omega)^2\sin^2(2\theta) + 4\sin(\theta)\sin(3\theta)\omega(B+\mu)$.

In order to analyze the (13) further, we make the following two cases.

Case(a): Let $\alpha > 2/3$ and $B+\mu > 0$. Then $|3\theta| = |\pm\frac{3\alpha\pi}{2}| > \pi$. Thus $\sin(3\theta)$ and $\sin(\theta)$ will have opposite signs. Similarly $\sin(3\theta)$ and $\sin(2\theta)$ will have opposite signs. Hence $\sin(3\theta)\sin(\theta) < 0$ and $\omega(B+\mu) < 0$ ($\omega < 0$). Hence $4\sin(\theta)\sin(3\theta)\omega(B+\mu) > 0$ which ensures that $\Delta > 0$.

Case(b): Let $\alpha < 2/3$ and $B+\mu < 0$. Then $|3\theta| = |\pm\frac{3\alpha\pi}{2}| < \pi$. Thus $\sin(3\theta)$, $\sin(\theta)$ and $\sin(2\theta)$ will have same signs. So $\sin(3\theta)\sin(\theta) > 0$ and $\omega(B+\mu) > 0$ ($\omega < 0$). Hence $4\sin(\theta)\sin(3\theta)\omega(B+\mu) > 0$ which will ensure $\Delta > 0$.

In view of (12), in both cases (a) and (b), we get

$$r_1 . r_2 = \frac{-\omega(B+\mu)\sin(\theta)}{\sin(3\theta)} < 0. \tag{14}$$

Eq. (14) implies that one of r_1,r_2 must be positive. Thus under these conditions we are assured of one real positive root for (12). Corresponding roots of (8) are given as

$$\lambda_1 = r_{1,2}e^{i\frac{\pi\alpha}{2}}, \quad \lambda_2 = r_{1,2}e^{-i\frac{\pi\alpha}{2}}. \tag{15}$$

After substituting (13) in (12) and solving for A_h we get

$$\begin{aligned}2\mu\omega A_h = \frac{1}{8}[&16B\mu\omega - 4(B+\mu)\omega\csc(3\theta)\cos(\theta)\\ &(-(B+\mu-\omega)\sin(2\theta) \pm \sqrt{\Delta}) + 2(B+\mu-\omega)\\ &\cos(2\theta)\csc^2(3\theta)(-(B+\mu-\omega)\sin(2\theta) \pm \sqrt{\Delta})^2\\ &+ \cot(3\theta)\csc^2(3\theta)(-(B+\mu-\omega)\sin(2\theta) \pm \sqrt{\Delta})^3].\end{aligned} \tag{16}$$

Eq. (16) represents **Hopf Critical Curve(HCC)** in $A - B$ parameter space. For given value of α, points on these curves are exact points at which the system undergoes Hopf bifurcation. In case of integer order derivative, *i.e.* $\alpha = 1$, $\theta = \pm\frac{\pi}{2}$ Eq. (16) simplifies to

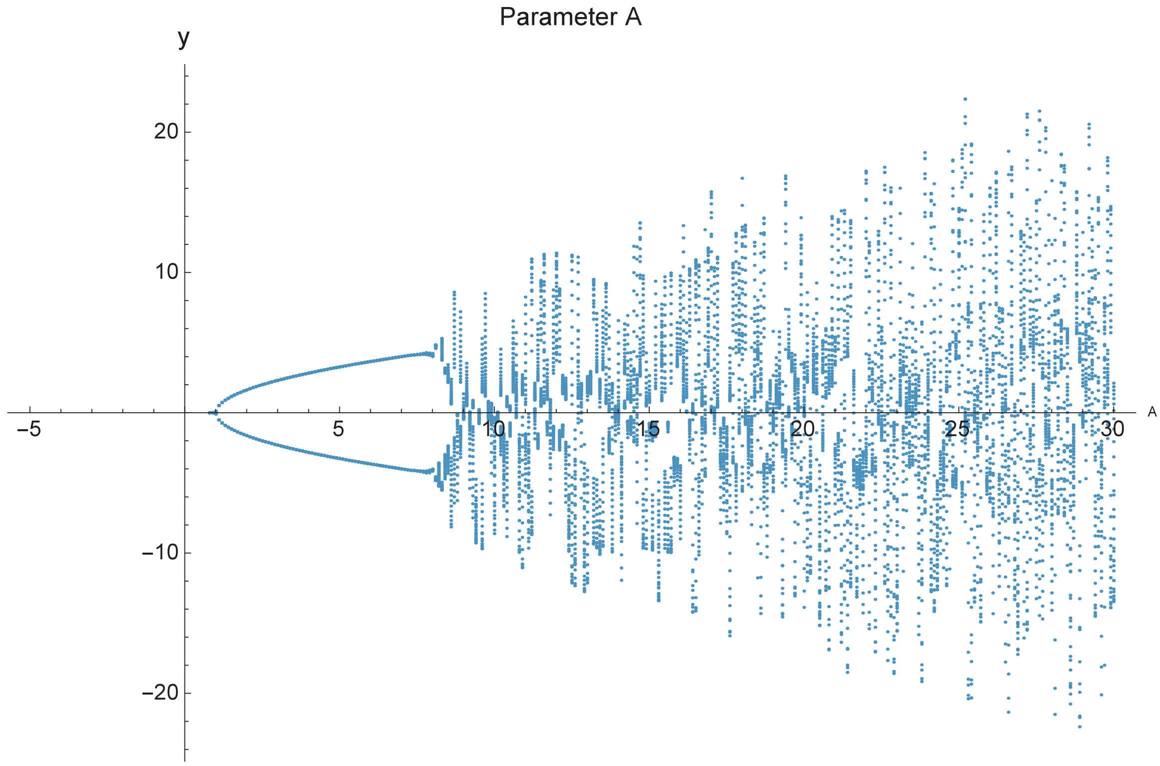
$$A_h = \frac{(B+\mu-\omega)(B+\mu) + 2\mu B}{2\mu}. \tag{17}$$

Theorem 1 (Existence of fractional Hopf bifurcation)**.** *Let $\mu > 0$, $\omega < 0$, and either $B+\mu > 0$, $\alpha > \frac{2}{3}$ or $B+\mu < 0$, $\alpha < \frac{2}{3}$, then the fractional BG system around equilibrium points P_2 and P_3 has one real eigenvalue say $\lambda_3(A)$ and two complex conjugate eigenvalues $\lambda_1(A), \lambda_2(A)$. Further there exists a Hopf critical value $A = A_h$ such that*

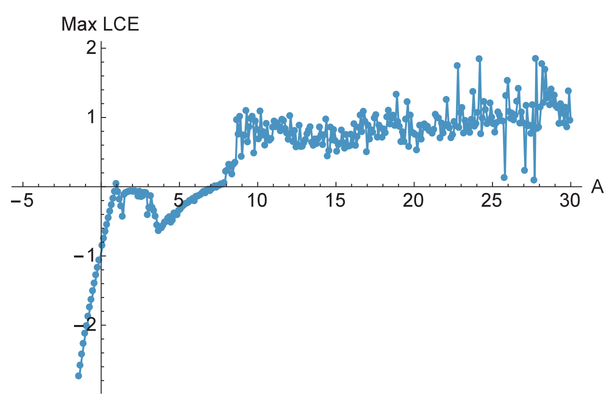
(i) *$\lambda_1(A_h)$ and $\lambda_2(A_h)$ satisfy $|arg(\lambda(A))| = \frac{\pi\alpha}{2}$*
(ii) *$\lambda_3(A_h) \neq 0$*
(iii) *$\frac{d}{dA}|arg(\lambda(A))|\,|_{A=A_h} \neq 0$ where $\lambda(A)$ refers to the complex eigenvalues.*

Proof. Existence of A_h such that $|\arg(\lambda_1(A_h))| = \frac{\pi\alpha}{2}$ and $|\arg(\lambda_2(A_h))| = \frac{\pi\alpha}{2}$ follows from (15) and (16).

For proving (ii), we assume $\lambda_3(A_h) = 0$. Then in view of (8), $2\mu\omega(B - A_h) = 0$. As $\mu,\ \omega \neq 0$, $B = A_h$. This is only single point in $A - B$ plane where Hopf critical curves meet $B = A$ line. This point lies on line $B = -\mu$ and by our assumption $B + \mu \neq 0$ which is a contradiction. Hence $\lambda_3(A_h) \neq 0$.



(a) Bifurcation diagram for the parameter A



(b) Maximum Lyapunov Characteristic Exponent (LCE) versus A plot.

Fig. 2. Bifurcation diagram and maximum LCE plot for parameter A with initial conditions as (0.1, 0.1, 0.1) and (−0.1, −0.1, −0.1) for $\alpha = 1$.

For proving (iii), we differentiate (8) with respect to A and obtain

$$\left[3\lambda^2 + 2\lambda(B+\mu-\omega) - \omega(B+\mu)\right]\frac{d\lambda}{dA} = 2\mu\omega. \tag{18}$$

Thus,

$$\frac{d\lambda}{dA} = \frac{2\mu\omega}{3\lambda^2 + 2(B+\mu-\omega)\lambda - \omega(B+\mu)}. \tag{19}$$

Then (15) implies

$$\left.\frac{d\lambda}{dA}\right|_{A=A_h} = \frac{2\mu\omega}{-\omega(B+\mu) + 2(B+\mu-\omega)r_{1,2}e^{i\theta} + 3r_{1,2}^2 e^{i2\theta}} = \frac{2\mu\omega(u-it)}{u^2+t^2}, \tag{20}$$

where $u = -\omega(B+\mu) + 2(B+\mu-\omega)r_{1,2}\cos(\theta) + 3r_{1,2}^2\cos(2\theta)$ and $t = 2(B+\mu-\omega)r_{1,2}\sin(\theta) + 3r_{1,2}^2\sin(2\theta)$. Let $\lambda(A) = p + iq$, then $\arg(\lambda(A)) = \arctan\left(\frac{q}{p}\right)$. By differentiating with respect to A, we get

$$\frac{d}{dA}\arg(\lambda(A)) = \frac{pq' - qp'}{p^2+q^2} = \frac{1}{|\lambda(A)|^2}W(p,q), \tag{21}$$

where $W(p,q) = \left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|$. Thus

$$\left.\frac{d}{dA}\arg(\lambda(A))\right|_{A=A_h} = \frac{1}{|\lambda(A_h)|^2}W(p,q)(A_h). \tag{22}$$

As $|\lambda(A_h)| < \infty$, it suffices to prove that $W(p,q)(A_h) \neq 0$.

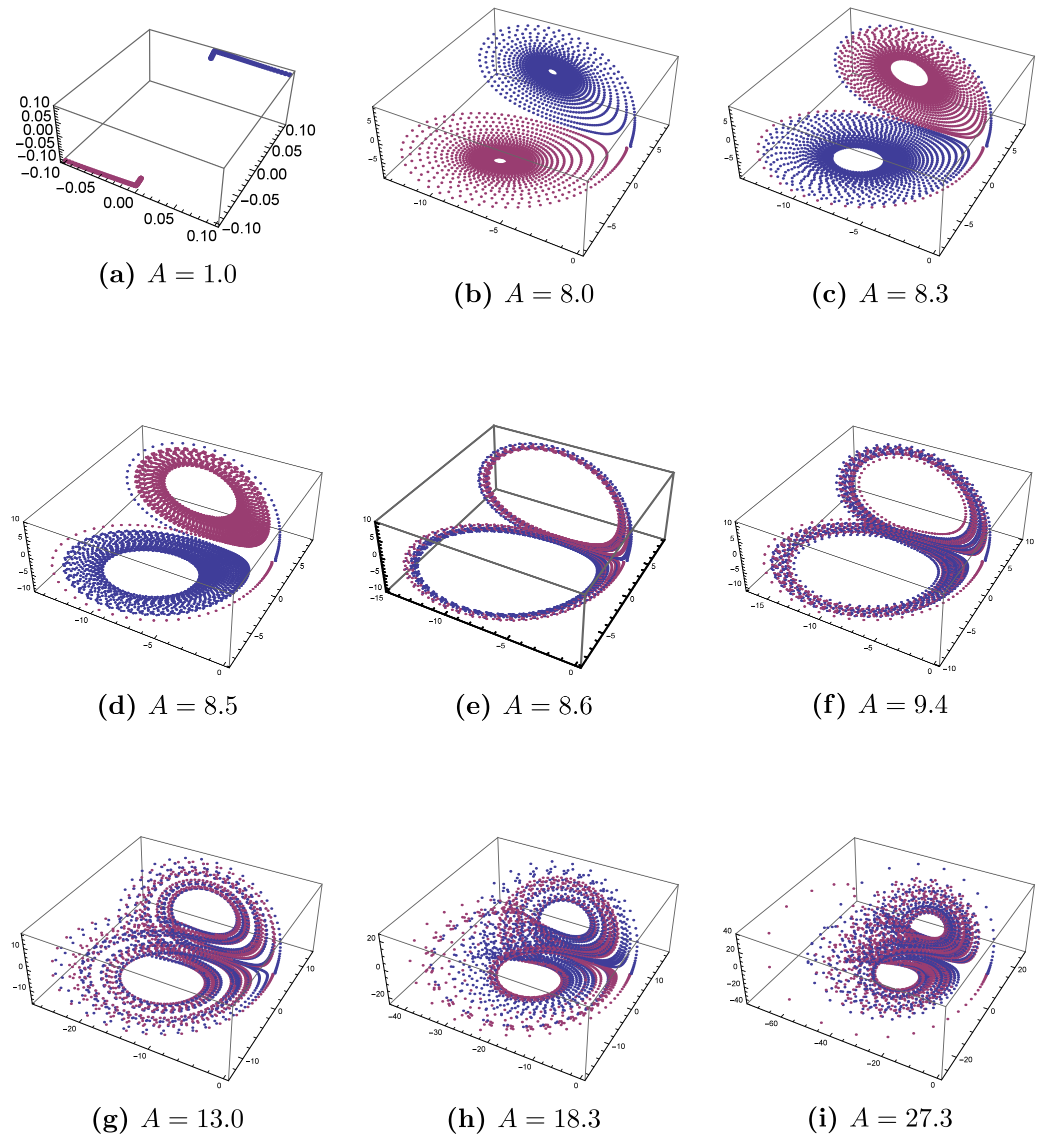


Fig. 3. Phase portraits of BG system with $\alpha = 1$ and initial conditions (0.1, 0.1, 0.1) and (−0.1, −0.1, −0.1) for certain values of A.

Suppose $W(p, q)(A_h) = 0$. From (15) we have $p(A_h) = r_{1,2}\cos(\theta)$ and $q(A_h) = r_{1,2}\sin(\theta)$. Also from (20) we have

$$p'(A_h) = \frac{2\mu\omega u}{u^2 + t^2}, \quad q'(A_h) = \frac{2\mu\omega t}{u^2 + t^2}. \tag{23}$$

In view of this

$$W(p, q)(A_h) = \begin{vmatrix} r_{1,2}\cos(\theta) & r_{1,2}\sin(\theta) \\ \frac{2\mu\omega u}{u^2+t^2} & \frac{2\mu\omega t}{u^2+t^2} \end{vmatrix},$$

$$= \frac{2\mu\omega}{u^2 + t^2}[-r_{1,2}\cos(\theta)t - r_{1,2}\sin(\theta)u]. \tag{24}$$

As $W(p, q)(A_h) = 0$, using (11) we get

$$-r_{1,2}\cos(\theta)t - r_{1,2}\sin(\theta)u = 0.$$

This implies

$$-3r_{1,2}^3[\sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta)] - 2r_{1,2}^2(B + \mu - \omega)$$
$$\times \sin(2\theta) + \omega(B + \mu)r_{1,2}\sin(\theta) = 0.$$

And hence

$$-\left[r_{1,2}^3\sin(3\theta) + r_{1,2}^2(B + \mu - \omega)\sin(2\theta) - \omega(B + \mu)r_{1,2}\sin(\theta)\right]$$
$$-2r_{1,2}^3\sin(3\theta) - r_{1,2}^2(B + \mu - \omega)\sin(2\theta) = 0.$$

This further simplifies as

$$-r_{1,2}^2[2r_{1,2}\sin(3\theta) + (B + \mu - \omega)\sin(2\theta)] = 0. \tag{25}$$

As $r_{1,2} \neq 0$ and $\sin(3\theta) \neq 0$ by rearranging (25) we get

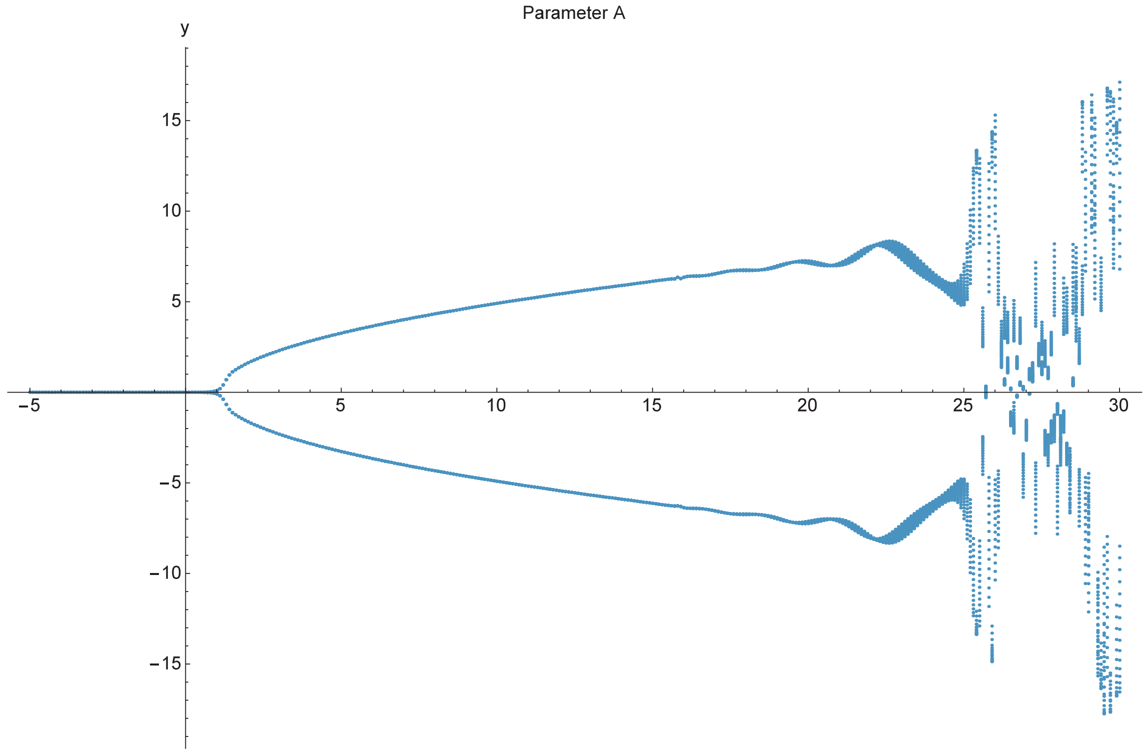
$$r_{1,2} = \frac{-(B + \mu - \omega)\sin(2\theta)}{2\sin(3\theta)}. \tag{26}$$

Comparing (13) and (26) we get $\Delta = 0$. This is a contradiction to the hypothesis $\Delta > 0$. Hence the result. □

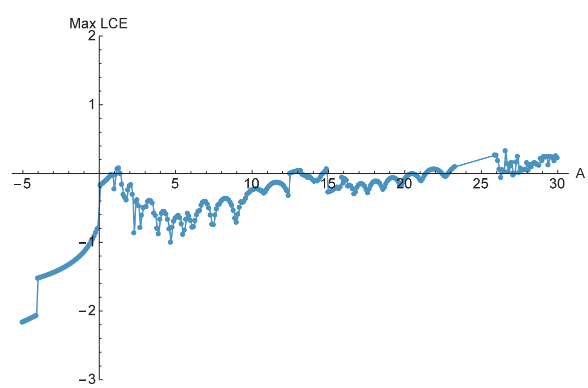
Consider **case(a):** Let $\omega = -2.667, \mu = 10$ fixed and $\alpha > 2/3$, $(B + \mu) > 0$. We plot the curves (16) in $A - B$ plane for $\alpha = 1, 0.92, 0.89, 0.86$ (cf. Fig. (1 a)). All the curves are parabolic in nature and are tangential to line $B = A$. Line $B = -\mu$ forms the lower boundary of the region due to the restriction $B + \mu > 0$. We observe that as α decreases, the slope of parabola also decreases.

Some remarks for the case(a) are in order:

Remark 1. For $B - A > 0$, only single equilibrium point exists and the system is stable. As we increase the value of the parameter A, at the line $B = A$, the system undergoes a pitchfork bifurcation. As per the value of α, fractional Hopf bifurcation will take place when parameter value crosses the corresponding HCC.



(a) Bifurcation diagram for the parameter A



(b) Maximum Lyapunov Characteristic Exponent (LCE) versus A plot.

Fig. 4. Bifurcation diagram and maximum LCE plot for parameter A with initial conditions as (0.1, 0.1, 0.1) and (−0.1, −0.1, −0.1) and $\alpha = 0.89$.

Remark 2. In case(a), the nature of the Hopf bifurcation is subcritical. This has been verified through numerical explorations in Sections 4 and 5 for $\alpha = 1$ and $\alpha = 0.89$ respectively. Due to subcritical Hopf bifurcation, system soon turns chaotic beyond HCC. Thus BG system shows chaotic behavior when parameter value (A, B) lies inside region between the line $B = -\mu$ and HCC.

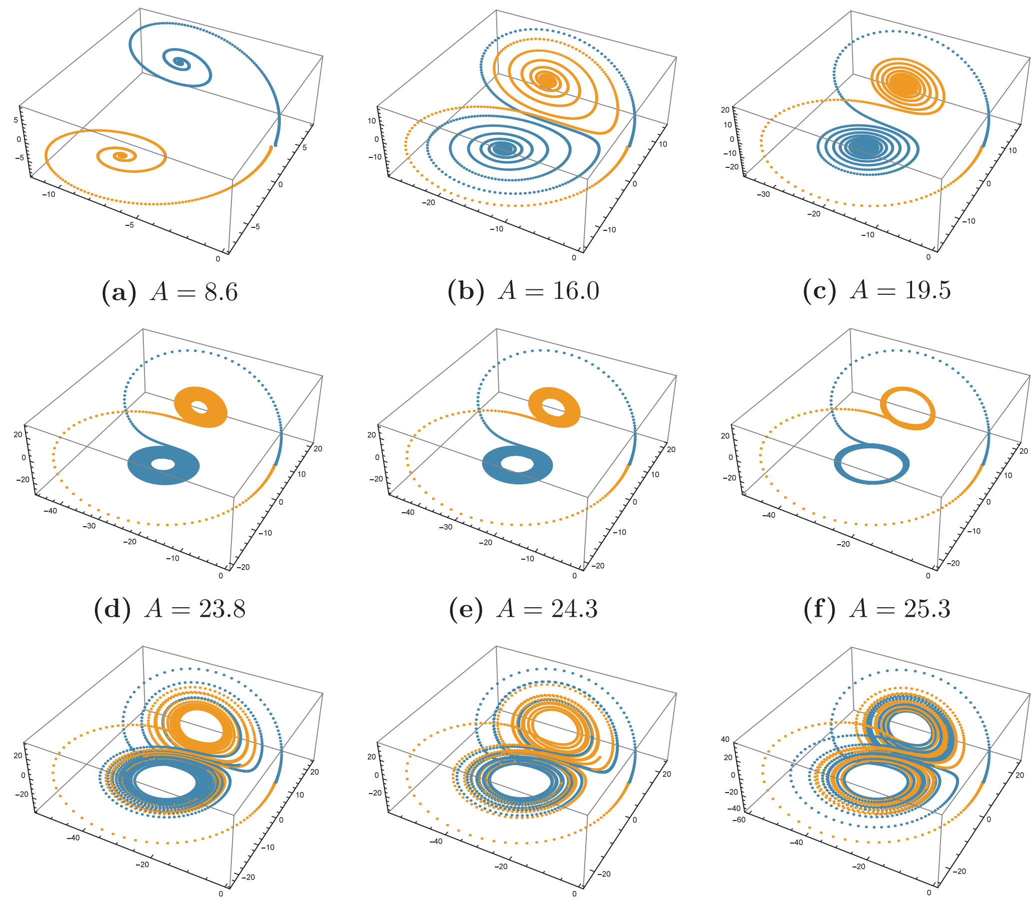
Remark 3. For $\alpha = 0.80$, the point $P(27.3, 1)$ lies to the exterior of the corresponding HCC and hence the system becomes stable at this threshold value, while for another point say $Q(40, 1)$ chaos would still be present for the same α. Thus we emphasize that for fractional systems, concept of threshold value is dependent on other system parameter values.

Case(b): Let $\omega = -2.667, \mu = 10$ and $\alpha < 2/3$, $(B + \mu) < 0$. Hopf Critical Curves (HCC) are plotted in $A - B$ plane for $\alpha = 0.10, 0.30, 0.40, 0.50$ (cf. Fig. (1 b)).

Some remarks are in order for the case(b):

Remark 4. For $B - A > 0$, only single equilibrium point exists and the system is stable. As we increase parameter A, at line $B = A$, the system undergoes a pitchfork bifurcation. According to the value of α, fractional Hopf bifurcation occurs when parameter value crosses the corresponding HCC.

Remark 5. Nature of the Hopf bifurcation is supercritical. This has been verified through numerical explorations in Section 6 for $\alpha = 0.65$. Line $B = -\mu$ forms an upper boundary for this region. System does not show chaos for any parameter values in this region.



(a) $A = 8.6$ (b) $A = 16.0$ (c) $A = 19.5$

(d) $A = 23.8$ (e) $A = 24.3$ (f) $A = 25.3$

(g) $A = 27.1$ (h) $A = 28.7$ (i) $A = 29.6$

Fig. 5. Phase portraits of BG system with $\alpha = 0.89$ and initial conditions $(0.1, 0.1, 0.1)$ and $(-0.1, -0.1, -0.1)$ for prominent values of A.

We define the global threshold value over (one or more) system parameters as follows.

Definition 8. We define global threshold value over parameter μ, (α^*_μ), of fractional system (1) as

$$\alpha^*_\mu = \inf_{\alpha_c}\{\text{for } \alpha > \alpha_c, \ \text{there exists } \mu = \mu_0, \ \text{such that system (1) is chaotic}\}. \quad (27)$$

Thus for any $\alpha < \alpha^*_\mu$, the fractional system will be stable regardless of the parameter value μ.

For fractional BG system with $\mu = 10$, $\omega = -2.667$ fixed, global threshold value over parameters A and B is $\alpha^*_{(A,B,10,-2.667)} = 2/3$.

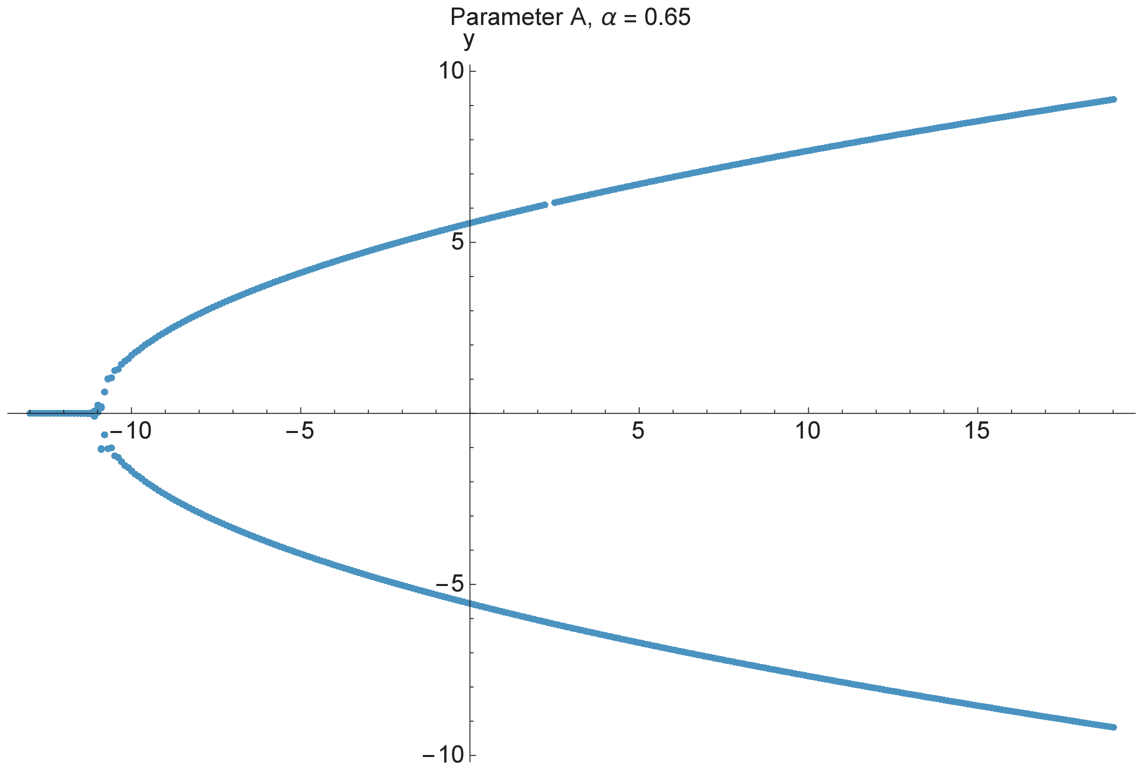
In further sections we verify the results numerically.

There are various numerical/ analytical methods for solving nonlinear fractional differential equations such as Fractional Adams method (FAM) [17], New Predictor Corrector Method (NPCM) [18], Adomian Decomposition Method (ADM) [19], New Iterative Method (NIM) [20], Frequency Domain Method and so on. Among these methods ADM or NIM give analytical solutions in the neighborhood of initial conditions. Hence these methods have limitations while finding long time behavior of solutions. It has been pointed out that frequency domain method is not always reliable for detection of chaos in fractional systems [21]. Hence for simulations pertaining to dynamical systems wherein long time behavior of the trajectories is studied, methods such as FAM or NPCM are more suitable. Hence while performing the numerical computations in the present article, we have used Fractional Adams Method (FAM) which is extensively used in the literature [17] along with **Mathematica 10.0** software.

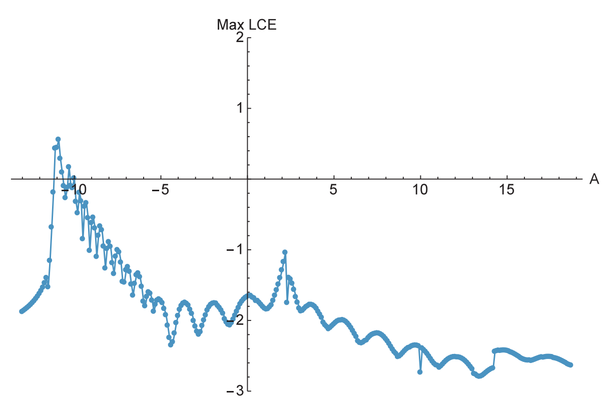
4. Numerical explorations for $\alpha = 1$

For $B = 1$, $\mu = 10$ and $\omega = -2.667$, the parameter A is varied in the interval $[-5, 30]$ in steps of 0.1 and the corresponding bifurcation diagram, maximum LCE plot and phase diagrams are studied for various values of A. For all the plots we use two sets of initial conditions *i.e.* $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$ and $(x_0, y_0, z_0) = (-0.1, -0.1, -0.1)$.

It is clear from Fig. (2) that the system becomes chaotic beyond the critical value $A_h = 8.5$. For $B - A = 0$, *i.e.* for $A < 1$, origin P_1, is the only equilibrium point of the system which is stable. As we increase A, at $A = 1$, system undergoes supercritical pitchfork bifurcation. Two new equilibrium points P_2 and P_3 start separating from the origin on $y = z$ plane, while at the same time origin loses its stability and turns into an index 1 saddle (cf. Fig. (3 a)). These new equilibrium points are stable attractors upto $A_h = 8.5$ (cf Fig. (3 c)). At $A_h = 8.5$ they undergo subcritical Hopf bifurcation. Due to this both P_2 and P_3 lose stability and turn into index-2 saddle points, and start repelling the trajectories. This intermixing of trajectories is visible in Fig. (3 e) and results into formation of chaotic attractor (cf. Fig. (3f) and(3g)). Fig. (3h) and (3i) show different stages in chaotic behavior.



(a) Bifurcation diagram for the parameter A



(b) Maximum Lyapunov Characteristic Exponent (LCE) versus A plot.

Fig. 6. Bifurcation diagram, maximum LCE plot for parameter A with initial conditions as $(0.1, 0.1, 0.1)$ and $(-0.1, -0.1, -0.1)$ and $\alpha = 0.65$ and $\alpha = 0.65$.

5. Numerical explorations for $\alpha = 0.89$

For $\mu = 10$, $\omega = -2.667$ and $B = 1$, parameter A is varied in the interval $[-5, 30]$. The corresponding bifurcation diagram and maximum LCE plots are presented in Fig. (4). It should be noted that Hopf critical value for the system is $A_h = 26.6$, whereas Hopf critical value for the corresponding integer order BG system is $A_h = 8.5$. It is noteworthy that for $A \in [8.53, 26.6]$ the integer order system shows chaotic behavior while fractional order system with $\alpha = 0.89$ still remains stable. The chaos ensues in $\alpha = 0.89$ system for $A > A_h$ (compare Fig. (2) and Fig. (4)).

In Fig. (5) phase diagrams for various values of A are drawn.

6. Numerical explorations for $\alpha = 0.65$

For $\mu = 10, \omega = -2.667$ and $B = -11$, parameter A is varied in the interval $[-13, 19]$. Corresponding bifurcation diagram and maximum LCE has been plotted in Fig. (6). It should be noted that for $A = -11$, the system undergoes supercritical pitchfork bifurcation, generating two new stable equilibrium points. At $A_h = -10.88$, the system undergoes supercritical Hopf bifurcation with formation of attracting limit cycles. (c.f. Fig. (7)). It is clear from Fig. (6) that system remains stable for the entire interval $[-13, 19]$. Fig. (8)shows phase diagrams for various values of $A \in [-13, 19]$.

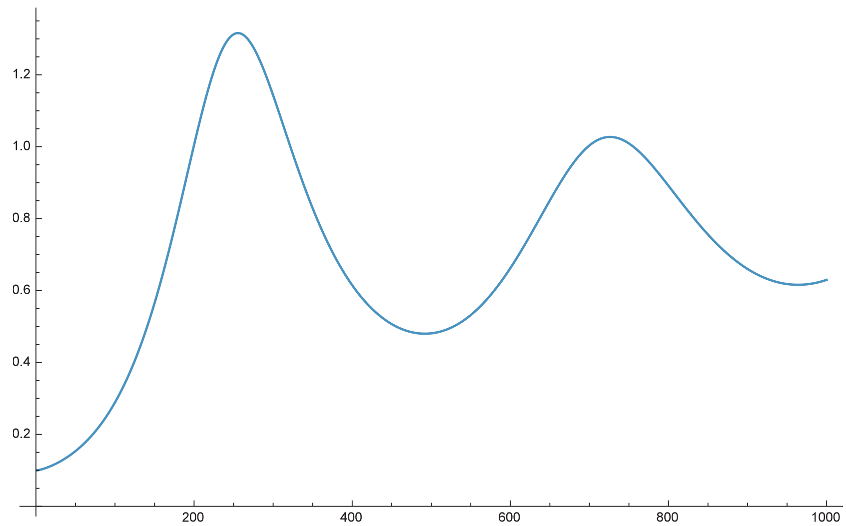


Fig. 7. Plot of $y(t)$ for $A = -10.8$. and $\alpha = 0.65$.

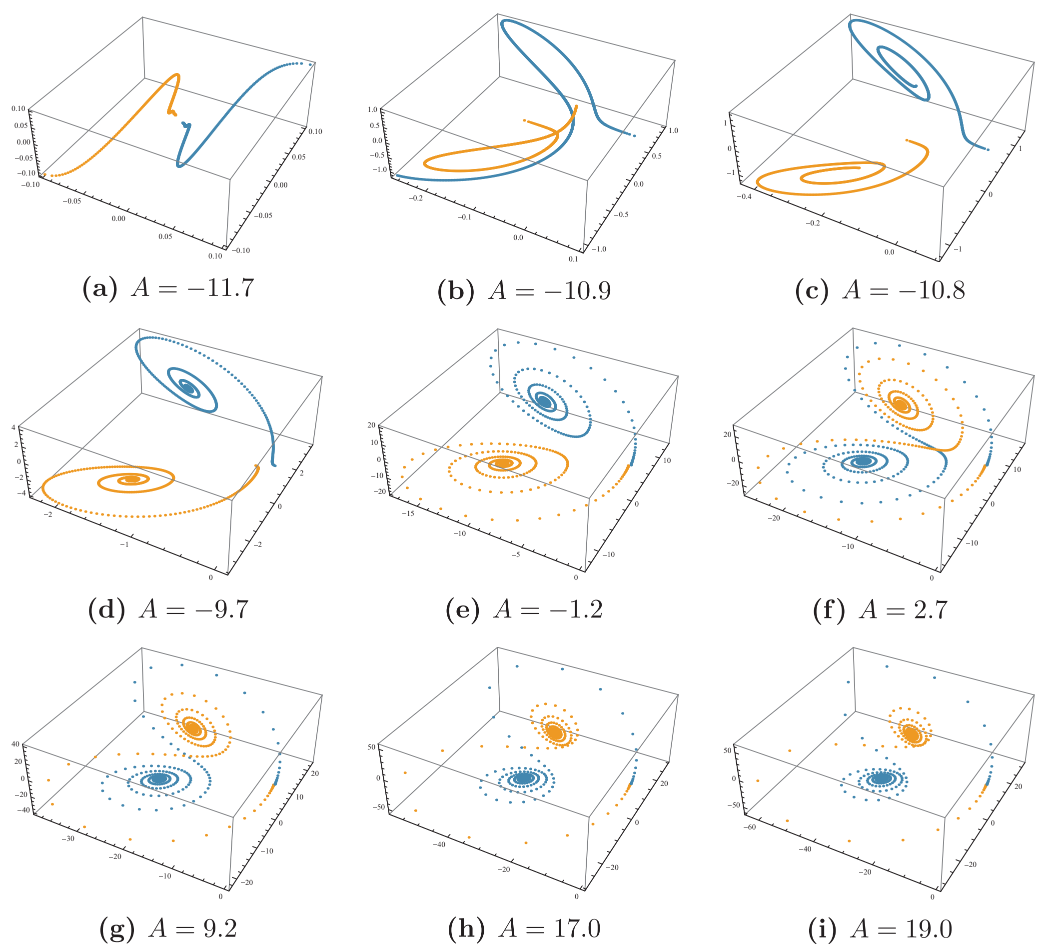


Fig. 8. Phase portraits of BG system for various values of A ($\alpha = 0.65$ and initial conditions (0.1, 0.1, 0.1) and (−0.1, −0.1, −0.1)).

7. Conclusions

Fractional Hopf bifurcation has been studied in case of fractional Bhalekar–Gejji system and the corresponding existence curves *i.e.* Hopf critical curves (HCC) have been plotted for various values of α, $0 < \alpha \leq 1$ in $A - B$ parameter space.

Exact regions in (A, B, α) parameter space have been identified for which the system is chaotic. Further global threshold value of the fractional order α over $A - B$ parameter space is found to be 2/3.

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