1. Problem statement

We want to compute the steady-state oscillations of the equation

$$
\begin{aligned}
&u_{tt} + c^2 u_{xxxx} = f(x)\cos\Omega t, \\
&u(0,t) = u_{xx}(0,t) = u(L,t) = u_{xx}(L,t) = 0.
\end{aligned}
\tag{1}
$$

2. The homogeneous equation

We first look at the homogeneous equation:

$$
\begin{aligned}
&u_{tt} + c^2 u_{xxxx} = 0, \\
&u(0,t) = u_{xx}(0,t) = u(L,t) = u_{xx}(L,t) = 0,
\end{aligned}
$$

and separate the variables by letting $u(x,t) = X(x)T(t)$. Thus,

$$X(x)T''(t) + c^2 X''''(x)T(t) = 0,$$

and therefore

$$-\frac{T''(t)}{T(t)} = c^2 \frac{X''''(x)}{X(x)} = \omega^2,$$

and we arrive at these two ordinary differential equations

$$T''(t) + \omega^2 T(t) = 0, \qquad X''''(x) + \left(\frac{\omega}{c}\right)^2 X(x) = 0.$$

For convenience, we define $\lambda = \sqrt{\frac{\omega}{c}}$, whereby the second equation takes the form $X'''' + \lambda^4 X = 0$. Then we solve for $T(t)$ and $X(x)$:

$$
\begin{aligned}
T(t) &= A_n \cos\omega t + B_n \sin\omega t, \\
X(x) &= c_1 \sin\lambda x + c_2 \cos\lambda x + c_3 \sinh\lambda x + c_4 \cosh\lambda x.
\end{aligned}
$$

The original PDE's boundary conditions translate to $X(0) = X''(0) = X(L) = X''(L) = 0$. Considering that

$$X''(x) = \lambda^2 \big[c_1 \sin\lambda x + c_2 \cos\lambda x + c_3 \sinh\lambda x + c_4 \cosh\lambda x\big],$$

we see that $X(0) = X''(0) = 0$ imply that $c_2 = c_4 = 0$, and thus we are left with $X(x) = c_1 \sin\lambda x + c_3 \sinh\lambda x$. Applying the boundary conditions $X(L) = X''(L) = 0$ yields

$$
\begin{aligned}
c_1 \sin\lambda L + c_3 \sinh\lambda L = 0, \\
c_1 \sin\lambda L - c_3 \sinh\lambda L = 0.
\end{aligned}
$$

Subtracting the two equations we get $c_3 \sinh\lambda L = 0$. We don't want $\lambda = 0$ since that gives us the trivial solution only. But if $\lambda \neq 0$, we have $\sinh\lambda L \neq 0$, and therefore $c_3 = 0$. Then either of the two equations above imply that $c_1 \sin\lambda L = 0$.

We don't what $c_1 = 0$ since that gives us the trivial solution only. Therefore $\sin\lambda L = 0$ which has infinitely many solutions

$$\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \dots.$$

Referring to the definition $\lambda = \sqrt{\frac{\omega}{c}}$, we denote the corresponding ω values by $\omega_n = c\lambda_n^2$.

The family of function $\{\sin \lambda_n x\}_{n=1}^{\infty}$ is an orthogonal basis for square-integrable functions on $(0, L)$ under the inner product

$$(f, g) = \int_0^L f(x)g(x)\, dx.$$

Therefore, any square-integrable function $f(x)$ may be expressed as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \lambda_n x, \tag{2}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x\, dx. \tag{3}$$

3. Steady-state oscillations

We return to the original statement of the problem in (1). We expand $f(x)$ into a sum of eigenfunctions as in (2), where the coefficients a_n may be computed through (3). We also expand the solution $u(x,t)$ into a sum of eigenfunctions

$$u(x,t) = \sum_{n=1}^{\infty} p_n(t) \sin \lambda_n x,$$

where the coefficients $p_n(t)$ are to be determined. Plug these into the PDE in (1). We get:

$$\sum_{n=1}^{\infty} p_n''(t) \sin \lambda_n x + c^2 \sum_{n=1}^{\infty} \lambda_n^4 p_n(t) \sin \lambda_n x = \Big(\sum_{n=1}^{\infty} a_n \sin \lambda_n x \Big) \cos \Omega t,$$

whence

$$p_n''(t) + c^2 \lambda_n^4 p_n(t) = a_n \cos \Omega t.$$

Recalling the definition of ω_n, this takes the form

$$p_n''(t) + \omega_n^2 p_n(t) = a_n \cos \Omega t.$$

Look for a particular solution of this ODE in the form $p(t) = K_n \cos \Omega t$. Plugging that form into the ODE we get

$$-\Omega^2 K_n + \omega_n^2 K_n = a_n,$$

whence

$$K_n = \frac{a_n}{\omega_n^2 - \Omega^2}.$$

We conclude that the steady-state oscillations of (1) are given by

$$u(x,t) = \Big(\sum_{n=1}^{\infty} \frac{a_n}{\omega_n^2 - \Omega^2} \sin \lambda_n x \Big) \cos \Omega t. \tag{4}$$

4. The special forcing function

The expression (4) gives the steady-state oscillations of the beam in (1) for any function $f(x)$. In particular if f is given as

$$f(x) = F_0\big(1 - \frac{x}{L}\big),$$

then from (3) we get

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\lambda_n x\,dx \\ &= \frac{2F_0}{L}\int_0^L \big(1 - \frac{x}{L}\big)\sin\lambda_n x\,dx \\ &= \Big(\frac{2F_0}{L}\Big)\Big(\frac{L}{n\pi}\Big) = \frac{2F_0}{n\pi}, \end{aligned}$$

and therefore

$$u(x,t) = \frac{2F_0}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n(\omega_n^2 - \Omega^2)}\sin\lambda_n x\right)\cos\Omega t.$$