THE SOLUTION OF THE POISSON PROBLEM WITH A ROBIN BOUNDARY CONDITION ON A DISK

ROUBEN ROSTAMIAN

ABSTRACT. We outline a general approach to solving the Poisson problem with a Robin boundary condition on a disk through Fourier series. Then we present a special case for illustration.

1. THE STATEMENT OF THE PROBLEM

We wish to solve this Poisson problem on a disk:

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + f(r,\theta) = 0 \qquad 0 < r < R, -\pi \le \theta < \pi, \tag{1a}$$

$$\Big[p(\theta)u(r,\theta) + q(\theta)\frac{\partial u}{\partial r}\Big]_{r=R} = g(\theta) \qquad -\pi \le \theta < \pi, \tag{1b}$$

where the given function $f(r,\theta)$, $p(\theta)$, $q(\theta)$, $g(\theta)$, and the solution $u(r,\theta)$ are expected to be 2π-periodic in θ. Thus, we expand $u(r,\theta)$ and $f(r,\theta)$ into Fourier series

$$u(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos n\theta + b_n(r)\sin n\theta, \tag{2a}$$

$$f(r,\theta) = \alpha_0(r) + \sum_{n=1}^{\infty} \alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta, \tag{2b}$$

where the coefficients $a_0(r)$, $a_n(r)$, and $b_n(r)$ are to be determined, and the coefficients $\alpha_0(r)$, $\alpha_n(r)$, and $\beta_n(r)$ are given by

$$\alpha_0(r) = \frac{1}{2\pi}\int_{\pi}^{\pi} f(r,\theta)\,d\theta, \tag{3a}$$

$$\alpha_n(r) = \frac{1}{\pi}\int_{\pi}^{\pi} f(r,\theta)\cos n\theta\,d\theta, \qquad n = 1, 2, \dots, \tag{3b}$$

$$\beta_n(r) = \frac{1}{\pi}\int_{\pi}^{\pi} f(r,\theta)\sin n\theta\,d\theta, \qquad n = 1, 2, \dots. \tag{3c}$$

Date: July 5, 2021.

2. Solving the PDE

Plugging the expressions (2) into the PDE (1a) we get

$$\frac{1}{r}\Big(ra_0'(r)\Big)' + \frac{1}{r}\sum_{n=1}^{\infty}\Big[\Big(ra_n'(r)\Big)'\cos n\theta + \Big(rb_n'(r)\Big)'\sin n\theta\Big] - \frac{n^2}{r^2}\sum_{n=1}^{\infty}\Big[a_n(r)\cos n\theta + b_n(r)\sin n\theta\Big] + \alpha_0(r) + \sum_{n=1}^{\infty}\Big[\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta\Big] = 0,$$

which we regroup as

$$\frac{1}{r}\Big(ra_0'(r)\Big)' + \alpha_0(r) + \sum_{n=1}^{\infty}\Big[\frac{1}{r}\Big(ra_n'(r)\Big)' - \frac{n^2}{r^2}a_n(r) + \alpha_n(r)\Big]\cos n\theta + \sum_{n=1}^{\infty}\Big[\frac{1}{r}\Big(rb_n'(r)\Big)' - \frac{n^2}{r^2}b_n(r) + \beta_n(r)\Big]\sin n\theta = 0.$$

We conclude that

$$\frac{1}{r}\Big(ra_0'(r)\Big)' + \alpha_0(r) = 0, \tag{4a}$$

$$\frac{1}{r}\Big(ra_n'(r)\Big)' - \frac{n^2}{r^2}a_n(r) + \alpha_n(r) = 0 \qquad n = 1, 2, \dots, \tag{4b}$$

$$\frac{1}{r}\Big(rb_n'(r)\Big)' - \frac{n^2}{r^2}b_n(r) + \beta_n(r) = 0 \qquad n = 1, 2, \dots. \tag{4c}$$

Considering that $\alpha_0(r)$, $\alpha_n(r)$, and $\beta_n(r)$ are known and given by (3), we solve the differential equations (4) to determine $a_0(r)$, $a_n(r)$, and $b_n(r)$. These will each come with undetermined constant coefficients which are to be determined by applying the boundary condition (1b).

Applying the boundary condition in the general case can be a messy affair since evaluating a term such as $p(\theta)u(r,\theta)$ requires multiplying two infinite series.

3. A special case

Here we completely solve a special case of our boundary value problem

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + 1 = 0 \qquad 0 < r < 1, -\pi \le \theta < \pi, \tag{5a}$$

$$(1+\cos^2\theta)u(r,\theta) + \frac{\partial u}{\partial r}\Big|_{r=1} = 0 \qquad -\pi \le \theta < \pi, \tag{5b}$$

which corresponds to $f(r,\theta) = 1$, $p(\theta) = 1 + \cos^2\theta$, $q(\theta) = 1$, $g(\theta) = 0$, $R = 1$.

From (3) we get

$$\alpha_0(r) = 1, \quad \alpha_n(r) = 0, \quad \beta_n(r) = 0,$$

and consequently the differential equations (4) reduce to

$$\frac{1}{r}\Big(ra_0'(r)\Big)' + 1 = 0, \tag{6a}$$

$$\frac{1}{r}\Big(ra_n'(r)\Big)' - \frac{n^2}{r^2}a_n(r) = 0 \qquad n = 1, 2, \ldots, \tag{6b}$$

$$\frac{1}{r}\Big(rb_n'(r)\Big)' - \frac{n^2}{r^2}b_n(r) = 0 \qquad n = 1, 2, \ldots, \tag{6c}$$

whose solutions are

$$\begin{aligned}
a_0(r) &= A_0 + \tilde{A}_0 \ln r - \frac{1}{4}r^2, \\
a_n(r) &= A_n r^n + \tilde{A}_n r^{-n} && n = 1, 2, \ldots, \\
b_n(r) &= B_n r^n + \tilde{B}_n r^{-n} && n = 1, 2, \ldots,
\end{aligned}$$

where the uppercase letters indicate generic constants. To have a finite solution at $r = 0$, we need to take $\tilde{A}_0$, $\tilde{A}_n$, and $\tilde{B}_n$ to be zero. We conclude that the general solution of (5a) is

$$u(r, \theta) = A_0 - \frac{1}{4}r^2 + \sum_{n=1}^{\infty} r^n \Big[A_n \cos n\theta + B_n \sin n\theta\Big].$$

To apply the boundary condition (5b), we calculate

$$\frac{\partial u}{\partial r} = -\frac{1}{2}r + \sum_{n=1}^{\infty} nr^{n-1} \Big[A_n \cos n\theta + B_n \sin n\theta\Big].$$

Then, the boundary condition (5b) yields

$$(1 + \cos^2\theta)\Big[A_0 - \frac{1}{4} + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta)\Big] \\ - \frac{1}{2} + \sum_{n=1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta) = 0.$$

To simplify that expression, we apply the trigonometric identities

$$\begin{aligned}
(1 + \cos^2\theta)\Big(A_0 - \frac{1}{4}\Big) - \frac{1}{2} &= \Big(\frac{3}{2}A_0 - \frac{7}{8}\Big) + \Big(\frac{1}{2}A_0 - \frac{1}{8}\Big)\cos 2\theta, \\
(1 + \cos^2\theta)\cos n\theta &= \frac{1}{4}\cos(n-2)\theta + \frac{3}{2}\cos n\theta + \frac{1}{4}\cos(n+2)\theta, \\
(1 + \cos^2\theta)\sin n\theta &= \frac{1}{4}\sin(n-2)\theta + \frac{3}{2}\sin n\theta + \frac{1}{4}\sin(n+2)\theta,
\end{aligned}$$

whereby the previous equation transforms into

$$\begin{aligned}
&\Big(\frac{3}{2}A_0 - \frac{7}{8}\Big) + \Big(\frac{1}{2}A_0 - \frac{1}{8}\Big)\cos 2\theta \\
&\qquad + \sum_{n=1}^{\infty} A_n\Big(\frac{1}{4}\cos(n-2)\theta + \frac{3}{2}\cos n\theta + \frac{1}{4}\cos(n+2)\theta\Big) \\
&\qquad + \sum_{n=1}^{\infty} B_n\Big(\frac{1}{4}\sin(n-2)\theta + \frac{3}{2}\sin n\theta + \frac{1}{4}\sin(n+2)\theta\Big) \\
&\qquad + \sum_{n=1}^{\infty} nA_n\cos n\theta + nB_n\sin n\theta = 0.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} A_n\cos(n-2)\theta \\
&\qquad = A_1\cos(-\theta) + A_2\cos(0\theta) + A_3\cos\theta + A_4\cos 2\theta + \sum_{n=5}^{\infty} A_n\cos(n-2)\theta \\
&\qquad = A_2 + (A_1 + A_3)\cos\theta + A_4\cos 2\theta + \sum_{n=3}^{\infty} A_{n+2}\cos n\theta, \\
&\sum_{n=1}^{\infty} A_n\cos n\theta = A_1\cos\theta + A_2\cos 2\theta + \sum_{n=3}^{\infty} A_n\cos n\theta, \\
&\sum_{n=1}^{\infty} A_n\cos(n+2)\theta = \sum_{n=3}^{\infty} A_{n-2}\cos n\theta,
\end{aligned}$$

and similarly for $B_n\sin(n-2)\theta$, $B_n\sin(n+2)\theta$, and $B_n\sin n\theta$. We continue our calculation to obtain

$$\begin{aligned}
&\Big(\frac{3}{2}A_0 - \frac{7}{8}\Big) + \Big(\frac{1}{2}A_0 - \frac{1}{8}\Big)\cos 2\theta \\
&\qquad + \frac{1}{4}\Big[A_2 + (A_1 + A_3)\cos\theta + A_4\cos 2\theta + \sum_{n=3}^{\infty} A_{n+2}\cos n\theta\Big] \\
&\qquad + \frac{3}{2}\Big[A_1\cos\theta + A_2\cos 2\theta + \sum_{n=3}^{\infty} A_n\cos n\theta\Big] \\
&\qquad + \frac{1}{4}\Big[\sum_{n=3}^{\infty} A_{n-2}\cos n\theta\Big] \\
&\qquad + \frac{1}{4}\Big[(B_3 - B_1)\sin\theta + B_4\sin 2\theta + \sum_{n=3}^{\infty} B_{n+2}\sin n\theta\Big] \\
&\qquad + \frac{3}{2}\Big[B_1\sin\theta + B_2\sin 2\theta + \sum_{n=3}^{\infty} B_n\sin n\theta\Big] \\
&\qquad + \frac{1}{4}\Big[\sum_{n=3}^{\infty} B_{n-2}\sin n\theta\Big] = 0.
\end{aligned}$$

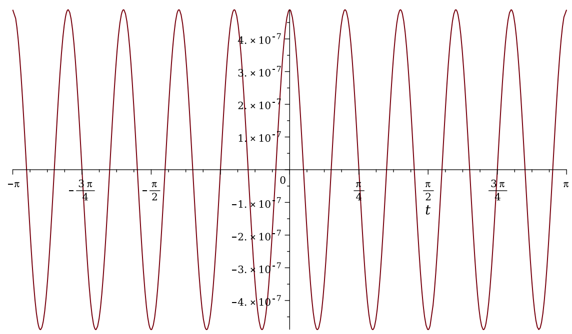

Figure 1. The residual error in the boundary condition (5b) corresponding to $N = 8$ terms in the Fourier series is $O(10^{-7})$.

Multiplying through by 4 and combining the terms, we arrive at the compact representation

$$\begin{aligned}6A_0 + A_2 - \frac{7}{2} + (11A_1 + A_3)\cos\theta + \Big(2A_0 + 14A_2 + A_4 - \frac{1}{2}\Big)\cos 2\theta \\ + (9B_1 + B_3)\sin\theta + (14B_2 + B_4)\sin 2\theta \\ + \sum_{n=3}^{\infty}\Big[\Big(A_{n-2} + 2(2n+3)A_n + A_{n+2}\Big)\cos n\theta \\ + \Big(B_{n-2} + 2(2n+3)B_n + B_{n+2}\Big)\sin n\theta\Big] = 0.\end{aligned}$$

To enforce the boundary condition, each of the coefficients in the Fourier series on the left-hand side of the equation above should be zero. That results in an infinitely many equations in the infinitely many unknowns A_k, B_k. Truncating the sums by replacing their upper limits with some number $N \geq 3$ results in a system of $5 + 2(N-2) = 2N + 1$ equations in $2N + 5$ unknowns $A_0, A_1, \dots A_N, A_{N+1}, A_{N+2}, B_1, \dots B_N, B_{N+1}, B_{N+2}$. To obtain a well-posed system, we set the coefficients $A_{N+1}, A_{N+2}, B_{N+1}, B_{N+2}$ to zero, and then solve the resulting system of $2N+1$ equations in $2N+1$ unknowns. The error in that substitution is expected to be small since the sequence of the coefficients of a Fourier series tends to zero. For instance, setting $N = 8$ we obtain

$$\begin{aligned}u(r,t) = 0.591 - 0.250r^2 - 0.0489r^2\cos 2\theta + 0.00223r^4\cos 4\theta \\ - 0.0000743r^6\cos 6\theta + 1.9610^{-6}r^8\cos 8\theta,\end{aligned}$$

which satisfies the PDE exactly, and the boundary condition approximately. The residual error in evaluating the boundary condition (5b) is of the order of magnitude of 10^{-7} as can be see in the graph in Figure 1. The corresponding solution is shown in Figure 2.

Department of Mathematics, UMBC, Baltimore, MD 21250
Email address: rostamian@umbc.edu

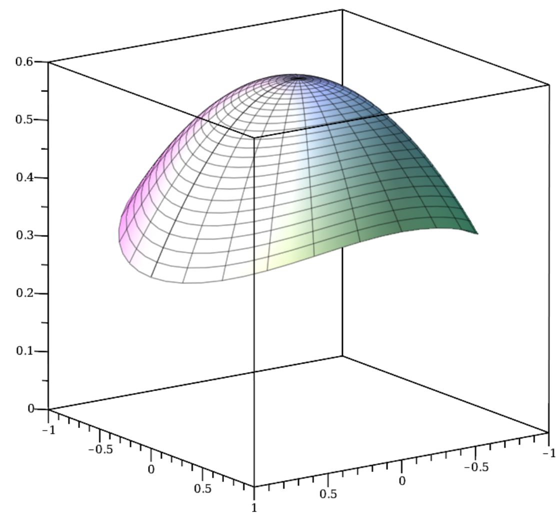

FIGURE 2. The solution of the boundary value problem (5) corresponding to $N = 8$ terms in the Fourier series.