THE STEADY-STATE SOLUTION OF A FIRST ORDER NON-AUTONOMOUS DE

ROUBEN ROSTAMIAN

Consider the initial value problem

$$x'(t) = p(t) - \big[a + q(t)\big]x(t), \quad x(0) = x_0, \tag{1}$$

where

(1) p and q are bounded periodic functions of some period, say σ;
(2) the average of q over a period is zero;
(3) $a > 0$ is a constant.

We show that the solution $x(t)$ converges to a σ-periodic steady-state. as $t \to \infty$. A particularly elementary instance of (1) is

$$x'(t) = 1 - \big[1 + \sin t\big]x(t).$$

Let $Q(t) = \int_0^t q(s)\,ds$. Then Q is also σ-periodic and $Q(0) = 0$. The differential equation (1) may be written as

$$x'(t) = p(t) - \big[a + Q'(t)\big]x(t), \quad x(0) = x_0, \tag{2}$$

The solution of (2) is obtained by the method of integrating factors:

$$x(t) = \left(x_0 + \int_0^t p(\xi)e^{a\xi + Q(\xi)}\,d\xi\right)e^{-at-Q(t)}. \tag{3}$$

We change the integration variable from ξ to $s = t - \xi$ and obtain

$$\begin{aligned} x(t) &= \left(x_0 + \int_0^t p(t-s)e^{a(t-s)+Q(t-s)}\,ds\right)e^{-at-Q(t)}. \\ &= x_0e^{-at-Q(t)} + \int_0^t p(t-s)e^{-as-Q(t)+Q(t-s)}\,ds. \end{aligned}$$

Let $T > 0$ be a sufficiently large time beyond which we expect the solution to be essentially periodic. Split the integration into integrals over the intervals $(0, T)$ and (T, t) to get

$$x(t) = x_0e^{-at-Q(t)} + \int_0^T p(t-s)e^{-as-Q(t)+Q(t-s)}\,ds + \int_T^t p(t-s)e^{-as-Q(t)+Q(t-s)}\,ds.$$

Since p and Q are σ-periodic in t, the integral over $(0, T)$ is σ-periodic in t. As to the integral over (T, t), it is the same as

$$\int_T^t \frac{p(t-s)e^{-Q(t)+Q(t-s)}}{e^{as}}\,ds.$$

We note that the integrand's numerator consists of periodic functions and is therefore bounded, while the denominator can be made arbitrarily large if T is sufficiently

Date: August 12, 2021.

large and $t > T$, in which case the integral would be essentially zero and may be discarded. We conclude that for $t > T$ and T sufficiently large we have

$$x(t) \approx x_0 e^{-at-Q(t)} + \int_0^T p(t-s) e^{-as-Q(t)+Q(t-s)}\, ds,$$

and thus, the transient and steady-state solutions are

$$x_{\mathrm{tr}}(t) = x_0 e^{-at-Q(t)}, \tag{4a}$$

$$x_{\mathrm{ss}}(t) = \int_0^T p(t-s) e^{-as-Q(t)+Q(t-s)}\, ds. \tag{4b}$$

See the corresponding Maple worksheet for examples.

Email address: `rostamian@umbc.edu`