Inner product using integrals

$a := 0$ $\quad b := 2$ $\quad f(x) := x^2 - 3x + 3$ $\quad g(x) := 2x^3 + x^2 + 5x - 5$ $\quad \int_a^b f(x)\, g(x)\, dx = 6$

The Legendre polynomials $\quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$

can be normalized on [-1,1] to $\quad \mathbb{P}_n = \sqrt{\frac{2n+1}{2}} \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$

Note that the Legendre polynomials are orthogonal. $\quad \int_{-b}^{b} \mathrm{Leg}(2,x)\, dx = 6$ $\quad \int_{-b}^{b} \mathrm{Leg}(2,x)\, \mathrm{Leg}(3,x)\, dx = 0$

When calculating the Legendre coefficients to use with Legendre polynomials to approximate functions, it is useful to calculate a weighting function that normalizes the polynomials.

$n := 0..5$ $\quad \mathrm{Leg}(n,0) = \begin{pmatrix} 1 \\ 0 \\ -0.5 \\ 0 \\ 0.375 \\ 0 \end{pmatrix}$ $\quad \int_{-1}^{1} P_n(x)\, P_m(n,x)\, dx = \frac{2}{2n+1} \delta(n,m)$ $\quad \sqrt{\frac{2n+1}{2}}\, \mathrm{Leg}(n,0) = \begin{pmatrix} 0.707 \\ 0 \\ -0.791 \\ 0 \\ 0.795 \\ 0 \end{pmatrix}$

Approximate f(x) using Legendre polynomials

$f(x) = \sum_n \left(c_n P_n(x)\right) \quad \Rightarrow \quad \left(P_m, f\right) = \sum_n \left[c_n\left(P_m, P_n\right)\right] = c_m \frac{2}{2n+1} \quad \Rightarrow \quad c_n = \frac{2n+1}{2}\left(P_n, f\right)$

where $\quad \left(P_m, P_n\right) = \frac{2}{2n+1} \delta(n,m)$

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$n := 0..9$ $\quad c_n := \frac{2n+1}{2} \int_{-1}^{1} f(x)\, \mathrm{Leg}(n,x)\, dx$ $\quad L(x) := \sum_n \left(c_n \mathrm{Leg}(n,x)\right)$

$c_n =$

0.500
0.750
0.000
-0.437
0.000
0.344
...