Computers and Mathematics with Applications 61 (2011) 2330–2341

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations

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ARTICLE INFO

Keywords:
Variational iteration method
Homotopy perturbation method
Boundary value problems
Integro-differential equations
Caputo fractional derivative

ABSTRACT

In this article, linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations are solved by variational iteration method and homotopy perturbation method. The fractional derivatives are described in the Caputo sense. The solutions of both problems are derived by infinite convergent series which are easily computable and then graphical representation shows that both methods are most effective and convenient one to solve linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations.

1. Introduction

In recent years various analytical and numerical methods have been applied for the approximate solutions of fractional differential equations (FDEs). Since exact solutions of most of the fractional differential equations do not exist, approximation and numerical methods are used for the solutions of the FDEs. He [1–4] was the first to propose the variational iteration method (VIM) and homotopy perturbation method (HPM) for finding the solutions of linear and nonlinear problems. VIM is based on Lagrange multiplier and HPM is a coupling of the traditional perturbation method and homotopy in topology. These methods have been successfully applied by many authors [3,5–10] for finding the analytical approximate solutions as well as numerical approximate solutions of functional equations which arise in scientific and engineering problems. The main feature for the use of VIM and HPM is that they can overcome the difficulties which arise in the Adomian decomposition method during computations of Adomian polynomials; see [8].

Many physical phenomena [11–15] can be modeled by fractional differential equations which have diverse applications in various physical processes such as acoustics, electromagnetism, control theory, robotics, viscoelastic materials, diffusion, edge detection, turbulence, signal processing, anomalous diffusion and fractured media. Momani and Aslam Noor [16] established the implementation of ADM to derive analytic approximate solutions of the linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations.

The purpose of this article is to extend the analysis of VIM and HPM to construct the approximate solutions of the following linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations

$$D^{\alpha}y(x) = f(x) + \gamma y(x) + \int_0^x [g(t)y(t) + h(t)F(y(t))]\mathrm{d}t \quad 0 < x < b,\ 3 < \alpha \leq 4 \tag{1}$$

subject to the following boundary conditions:

$$y(0) = \gamma_0, \qquad y''(0) = \gamma_2, \tag{2}$$

$$y(b) = \beta_0, \qquad y''(b) = \beta_2, \tag{3}$$

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doi:10.1016/j.camwa.2010.10.004

where D^{α} is the fractional derivative in the Caputo sense and $F(y(x))$ is any nonlinear function, γ, γ_0, γ_2, β_0 and β_2 are real constants and f, g and h are given and can be approximated by Taylor polynomials.

2. Basic definitions

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition 1. A real function $f(x)$, $x > 0$ is said to be in space C_{μ}, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^{p} f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_{μ}^{n} if and only if $f^{n} \in C_{\mu}$, $n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad \alpha > 0 \tag{4}$$
$$J^{0} f(t) = f(t).$$

Some properties of the operator J^{α}, which are needed here, are as follows:
For $f \in C_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$:

(1) $J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$
(2) $J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t)$
(3) $J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{d}s, \tag{5}$$

for $m - 1 < \alpha \leq m$, $m \in N$, $t > 0$, $f \in C_{-1}^{m}$.

Lemma 1. *If $m - 1 < \alpha \leq m$, $m \in N$, $f \in C_{\mu}^{m}$, $\mu \geq -1$, then the following two properties hold:*
(1) $D^{\alpha} J^{\alpha} f(t) = f(t)$
(2)

$$(J^{\alpha} D^{\alpha}) f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \frac{t^k}{k!}. \tag{6}$$

3. Analysis of VIM

To illustrate the basic concepts of variational iteration method, consider the fractional differential equation (1) with boundary conditions (2)–(3).

According to the variational iteration method, we can construct the correction functional for Eq. (1) as:

$$\begin{aligned} y_{k+1}(x) &= y_k(x) + J^{\beta}\left[\lambda\left(D^{\alpha} y_k(x) - f(x) - \gamma \tilde{y}_k(x) - \int_0^x [g(p)\tilde{y}_k(p) + h(p)F(\tilde{y}_k(p))]\mathrm{d}p\right)\right] \\ &= y_k(x) + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \lambda(s) \left(D^{\alpha} y_k(s) - f(s) - \gamma \tilde{y}_k(s) - \int_0^s [g(p)\tilde{y}_k(p) + h(p)F(\tilde{y}_k(p))]\mathrm{d}p\right) \mathrm{d}s \end{aligned} \tag{7}$$

where J^{β} is the Riemann–Liouville fractional integral operator of order $\beta = \alpha + 1 - m$, λ is a general Lagrange multiplier and $\tilde{y}_k$ denotes restricted variation i.e. $\delta \tilde{y}_k = 0$.

We make some approximation for the identification of an approximate Lagrange multiplier, so the correctional functional (7) can be approximately expressed as:

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda(s) \left(D^{4} y_k(s) - f(s) - \gamma \tilde{y}_k(s) - \int_0^s [g(p)\tilde{y}_k(p) + h(p)F(\tilde{y}_k(p))]\mathrm{d}p\right) \mathrm{d}s. \tag{8}$$

Making the above correction functional stationary, we obtain the following stationary conditions:

$$1 - \lambda'''(s)|_{s=x} = 0, \qquad \lambda''(s)|_{s=x} = 0,$$
$$-\lambda'(s)|_{s=x} = 0, \qquad \lambda(s)|_{s=x} = 0, \qquad \lambda^{(\mathrm{iv})}(s) = 0.$$

This gives the following Lagrange multiplier

$$\lambda(s) = \frac{1}{6}(s-x)^3. \tag{9}$$

We obtain the following iteration formula by substitution of (9) into functional (7),

$$y_{k+1}(x) = y_k(x) + \frac{1}{6\Gamma(\alpha-3)}\int_0^x (x-s)^{\alpha-4}(s-x)^3\left(D^\alpha y_k(s) - f(s) - \gamma y_k(s) - \int_0^s [g(p)y_k(p) + h(p)F(y_k(p))]\mathrm{d}p\right)\mathrm{d}s$$
$$= y_k(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(D^\alpha y_k(s) - f(s) - \gamma y_k(s) - \int_0^s [g(p)y_k(p) + h(p)F(y_k(p))]\mathrm{d}p\right)\mathrm{d}s.$$

This yields the following iteration formula:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6}J^\alpha\left(D^\alpha y_k(x) - f(x) - \gamma y_k(x) - \int_0^x [g(p)y_k(p) + h(p)F(y_k(p))]\mathrm{d}p\right). \quad (10)$$

The initial approximation y_0 can be chosen by the following way which satisfies initial conditions (2):

$$y_0(x) = \gamma_0 + \gamma_1 x + \frac{\gamma_2}{2}x^2 + \frac{\gamma_3}{6}x^3 \quad (11)$$

where $\gamma_1 = y'(0)$ and $\gamma_3 = y'''(0)$ are to be determined by applying suitable boundary conditions (3).

We can obtain the following first-order approximation by substitution of (11) into (10):

$$y_1(x) = y_0(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6}J^\alpha\left(D^\alpha y_0(x) - f(x) - \gamma y_0(x) - \int_0^x [g(p)y_0(p) + h(p)F(y_0(p))]\mathrm{d}p\right). \quad (12)$$

Similarly, we can obtain the higher-order approximations. If Nth-order approximate is enough, then imposing boundary conditions (3) in Nth-order approximation yields the following system of equations:

$$y_N(b) = \beta_0, \quad (13)$$
$$y''_N(b) = \beta_2. \quad (14)$$

From Eqs. (13)–(14), we can find the unknowns $\gamma_1 = y'(0)$ and $\gamma_3 = y'''(0)$. Substituting the constant values of γ_1 and γ_3 in Nth-order approximation results the approximate solution of (1)–(3).

4. Analysis of HPM

To illustrate the basic concepts of HPM for fractional integro-differential equations, consider the fractional differential equation (1) with boundary conditions (2)–(3).

In view of HPM [3,4], construct the following homotopy for Eq. (1):

$$(1-p)D^\alpha y(x) + p\left(D^\alpha y(x) - f(x) - \gamma y(x) - \int_0^x [g(t)y(t) + h(t)F(y(t))]\mathrm{d}t\right) = 0 \quad (15)$$

or

$$D^\alpha y(x) = p\left(f(x) + \gamma y(x) + \int_0^x [g(t)y(t) + h(t)F(y(t))]\mathrm{d}t\right) \quad (16)$$

where $p \in [0, 1]$ is an embedding parameter. If $p = 0$, then Eq. (16) becomes a linear equation,

$$D^\alpha y(x) = 0, \quad (17)$$

and when $p = 1$, then Eq. (16) turns out to be the original Eq. (1).

In view of basic assumption of homotopy perturbation method, solution of Eq. (1) can be expressed as a power series in p:

$$y(x) = y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x) + \cdots. \quad (18)$$

Setting $p = 1$ in (18) results the approximate solution of Eq. (1):

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots. \quad (19)$$

The convergence of series (19) has been proved in [17]. Substitution (18) into (16), then equating the terms with identical power of p, we obtain the following series of linear equations:

$$p^0 : D^\alpha y_0 = 0, \tag{20}$$

$$p^1 : D^\alpha y_1 = f(x) + \gamma y_0(x) + \int_0^x [g(t)y_0(t) + h(t)F_1(y_0(t))]\mathrm{d}t, \tag{21}$$

$$p^2 : D^\alpha y_2 = \gamma y_1(x) + \int_0^x [g(t)y_1(t) + h(t)F_2(y_1(t))]\mathrm{d}t, \tag{22}$$

$$p^3 : D^\alpha y_3 = \gamma y_2(x) + \int_0^x [g(t)y_2(t) + h(t)F_3(y_2(t))]\mathrm{d}t, \tag{23}$$

$$\vdots$$

where the functions $F_1, F_2, \ldots$ satisfy the following condition:

$$F(y_0(t) + py_1(t) + p^2y_2(t) + \cdots) = F_1(y_0(t)) + pF_2(y_1(t)) + p^2F_3(y_2(t)) + \cdots.$$

From Eq. (20), the initial approximation can be chosen in the following way:

$$y_0 = \sum_{j=0}^{3} \gamma_j \frac{x^j}{j!}$$

where $\gamma_1 = y'(0)$ and $\gamma_3 = y'''(0)$ are to be determined by applying suitable boundary conditions (3).

Eqs. (20)–(23) can be solved by applying the operator J^α, which is the inverse of the operator D^α and then by simple computation, we approximate the series solution of HPM by the following N-term truncated series:

$$\theta_N(x) = y_0(x) + y_1(x) + y_2(x) + \cdots + y_{N-1}(x). \tag{24}$$

Note that in expression (24), constants γ_1 and γ_3 are undetermined. By imposing boundary conditions (3) in (24), we get the following system of equations

$$y_0(b) + y_1(b) + y_2(b) + \cdots + y_{N-1}(b) = \beta_0, \tag{25}$$

$$y_0''(b) + y_1''(b) + y_2''(b) + \cdots + y_{N-1}''(b) = \beta_2. \tag{26}$$

From Eqs. (25)–(26), we can find the unknowns γ_1 and γ_3. Substituting the constant values of γ_1 and γ_3 in expression (24) results the approximate solution of (1)–(3).

5. Applications

In this section we have applied variational iteration method and homotopy perturbation method to fourth-order linear and nonlinear fractional integro-differential equations with a known exact solution at $\alpha = 4$.

Example 1. Consider the following linear fourth-order fractional integro-differential equation:

$$D^\alpha y(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)\mathrm{d}t \quad 0 < x < 1,\ 3 < \alpha \le 4 \tag{27}$$

subject to the following boundary conditions:

$$y(0) = 1, \qquad y''(0) = 2, \tag{28}$$

$$y(1) = 1 + e, \qquad y''(1) = 3e. \tag{29}$$

For $\alpha = 4$, the exact solution of problem (27)–(29) is

$$y(x) = 1 + xe^x.$$

According to variational iteration method, the iteration formula (10) for Eq. (27) can be expressed in the following form:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6} J^\alpha \left(D^\alpha y_k(x) - x(1+e^x) - 3e^x - y_k(x) + \int_0^x y_k(t)\mathrm{d}t \right). \tag{30}$$

In order to avoid difficult fractional integration, we can take the truncated Taylor expansion for the exponential term in (30): e.g., $e^x \sim 1 + x + x^2/2 + x^3/6$ and assume that an initial approximation has the following form which satisfies the initial conditions (28):

$$y_0(x) = 1 + Ax + x^2 + \frac{B}{6}x^3 \tag{31}$$

where $A = y'(0)$ and $B = y'''(0)$ are unknowns to be determined.

Table 1
Values of A and B for different values of α using (32).

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4$
A	0.74031475165214	0.81642134845857	0.90761047783198	0.99822354588777
B	5.40426563043794	4.54507997600139	3.71105498995859	3.01192914529881

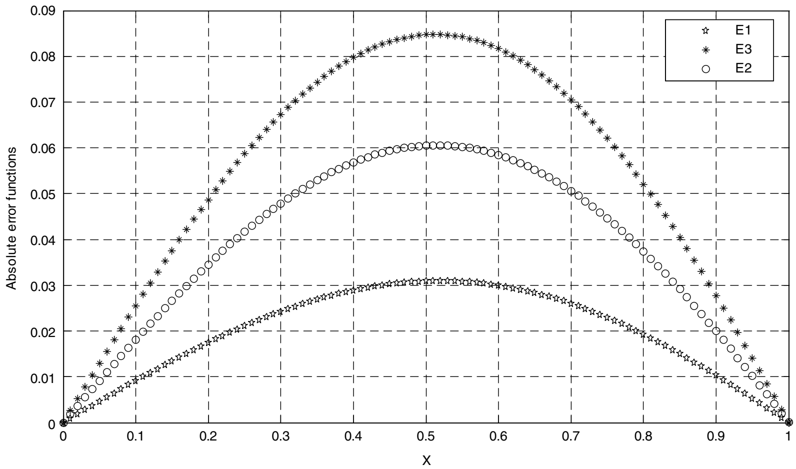

Fig. 1. Absolute error functions $E_1(x)$, $E_2(x)$ and $E_3(x)$ obtained by VIM with different values of α.

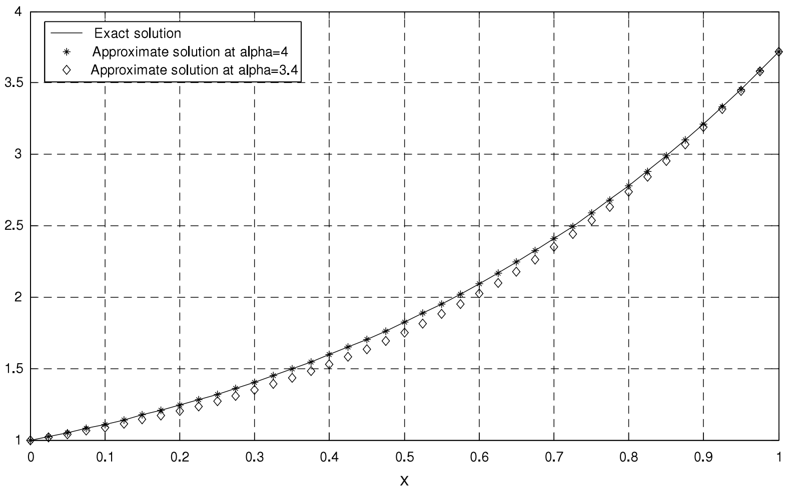

Fig. 2. Comparison of the first-order approximate solution obtained by VIM with the exact solution at $\alpha = 4$ and $\alpha = 3.4$.

Now, by iteration formula (30), first-order approximation takes the following form:

$$\begin{aligned} y_1(x) &= y_0(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6} J^\alpha \left(D^\alpha y_0(x) - 3 - 5x - \frac{5}{2}x^2 - x^3 - \frac{x^4}{6} - y_0(x) + \int_0^x y_0(t)\mathrm{d}t \right) \\ &= 1 + Ax + x^2 + \frac{B}{6}x^3 - \frac{(\alpha-3)(\alpha-2)(\alpha-1)x^\alpha}{6} \\ &\quad \times \left(-\frac{4}{\Gamma(\alpha+1)} - \frac{(4+A)x}{\Gamma(\alpha+2)} + \frac{(A-7)x^2}{\Gamma(\alpha+3)} - \frac{(4+B)x^3}{\Gamma(\alpha+4)} + \frac{(B-4)x^4}{\Gamma(\alpha+5)} \right). \end{aligned} \tag{32}$$

By imposing boundary conditions (29) in $y_1(x)$, we obtain Table 1 which shows the values of A and B for different values of α.

In Fig. 1, we draw absolute error functions $E_1(x) = |(1 + xe^x) - y_{1,3.75}|$, $E_2(x) = |(1 + xe^x) - y_{1,3.5}|$ and $E_3(x) = |(1 + xe^x) - y_{1,3.25}|$ for different values of α, where $1 + xe^x$ is an exact solution of (27)–(29) and $y_{1,3.75}$, $y_{1,3.5}$ and $y_{1,3.25}$ represent the values of $y_1(x)$ at $\alpha = 3.75$, $\alpha = 3.5$ and $\alpha = 3.25$, respectively (see Fig. 2).

Table 2
Values of A and B for different values of α using (43).

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4$
A	1.10186984200028	1.09179499393439	1.05222793297923	0.99906052231083
B	−0.29428456125416	0.96679305229906	2.07168041387465	3.00628128299199

According to HPM, we construct the following homotopy:

$$D^{\alpha}y(x) = p\left(x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)\mathrm{d}t\right). \tag{33}$$

Substitution of (18) into (33) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0 : D^{\alpha}y_0 = 0, \tag{34}$$

$$p^1 : D^{\alpha}y_1 = x(1+e^x) + 3e^x + y_0(x) - \int_0^x y_0(t)\mathrm{d}t, \tag{35}$$

$$p^2 : D^{\alpha}y_2 = y_1(x) - \int_0^x y_1(t)\mathrm{d}t, \tag{36}$$

$$p^3 : D^{\alpha}y_3 = y_2(x) - \int_0^x y_2(t)\mathrm{d}t, \tag{37}$$

$$\vdots$$

Applying the operator J^{α} to the above series of linear equations and using initial conditions (28), we get;

$$y_0(x) = 1, \tag{38}$$

$$y_1(x) = Ax + x^2 + \frac{1}{6}Bx^3 + J^{\alpha}\left(x(1+e^x) + 3e^x + y_0(x) - \int_0^x y_0(t)\mathrm{d}t\right), \tag{39}$$

$$y_n(x) = J^{\alpha}\left(y_{n-1}(x) - \int_0^x y_{n-1}(t)\mathrm{d}t\right), \quad n = 2, 3, 4, \ldots \tag{40}$$

where $A = y'(0)$ and $B = y'''(0)$ are to be determined.

In order to avoid difficult fractional integration, we can take the truncated Taylor expansions for the exponential term in (39)–(40): e.g., $e^x \sim 1 + \mathrm{x} + \mathrm{x}^2/2 + x^3/6$.

Thus, by solving Eqs. (38)–(40), we obtain $y_1, y_2, \ldots$ e.g.:

$$y_1(x) = Ax + x^2 + \frac{Bx^3}{6} + \frac{4x^{\alpha}}{\Gamma(\alpha+1)} + \frac{4x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{5x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{6x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)} \tag{41}$$

$$\begin{aligned} y_2(x) = {} & \frac{Ax^{\alpha+1}}{\Gamma(\alpha+2)} + (2-A)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + (B-2)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{Bx^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{4x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} \\ & + \frac{x^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{2x^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{4x^{2\alpha+5}}{\Gamma(2\alpha+6)}. \end{aligned} \tag{42}$$

Now, we can form the 2-term approximation

$$\begin{aligned} \phi_2(x) = {} & 1 + Ax + x^2 + \frac{Bx^3}{6} + \frac{4x^{\alpha}}{\Gamma(\alpha+1)} + (4+A)\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (7-A)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + (4+B)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)} \\ & + (4-B)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{4x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{x^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{2x^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{4x^{2\alpha+5}}{\Gamma(2\alpha+6)} \end{aligned} \tag{43}$$

where A and B can be determined by imposing boundary conditions (29) on ϕ_2. Table 2 shows the values of A and B for different values of α. In Fig. 3, we draw absolute error functions $E_4(x) = |(1+xe^x) - \phi_{2,3.75}|$, $E_5(x) = |(1+xe^x) - \phi_{2,3.5}|$ and $E_6(x) = |(1+xe^x) - \phi_{2,3.25}|$ for different values of α, where $1 + xe^x$ is an exact solution of (27)–(29) and $\phi_{2,3.75}$, $\phi_{2,3.5}$ and $\phi_{2,3.25}$ represent the values of ϕ_2 at $\alpha = 3.75$, $\alpha = 3.5$ and $\alpha = 3.25$, respectively.

In Figs. 4a–4c, we compare the approximate solutions obtained by VIM and HPM with an exact solution, and it is clear from Figs. 4a–4c that the approximate solutions are in good agreement with an exact solution of (27)–(29) at $\alpha = 4$, $\alpha = 3.8$ and $\alpha = 3.2$. Also it is to be noted that the accuracy can be improved by computing more terms of approximated solutions and/or by taking more terms in the Taylor expansion for the exponential term.

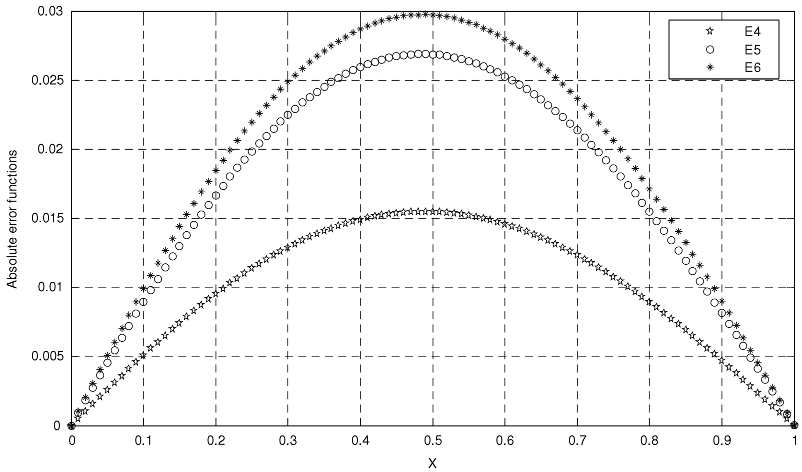

Fig. 3. Absolute error functions $E_4(x)$, $E_5(x)$ and $E_6(x)$ obtained by 2-term HPM with $\alpha = 3.25$, $\alpha = 3.5$ and $\alpha = 3.75$.

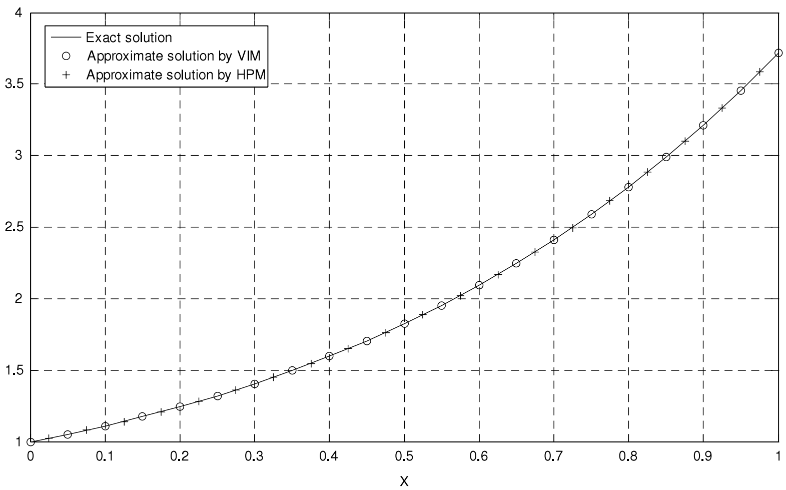

Fig. 4a. Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 4$.

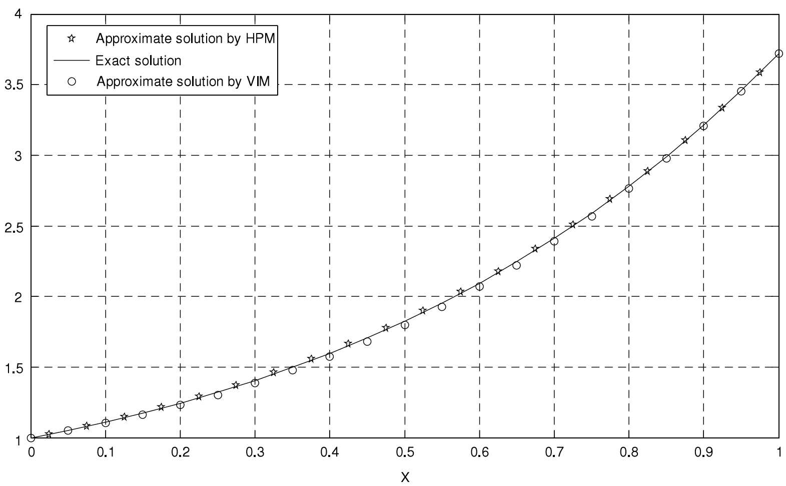

Fig. 4b. Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 3.8$.

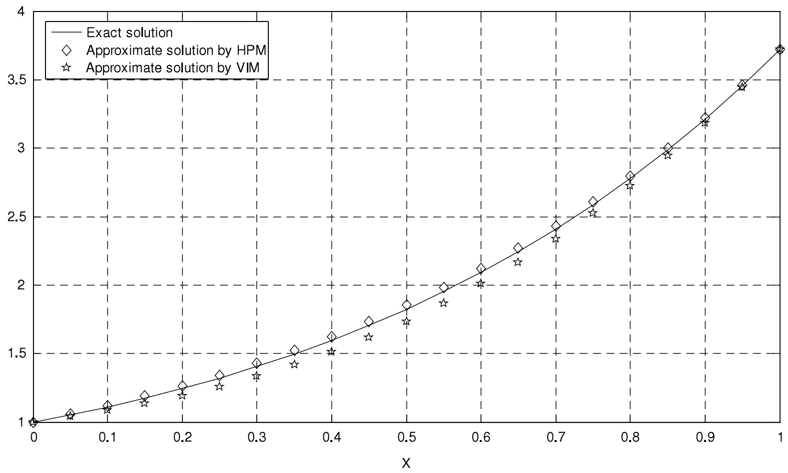

Fig. 4c. Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 3.2$.

Table 3
Values of A and B for different values of α using (49).

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4$
A	0.94142433289801	0.95800824448441	0.97872080488492	0.99983099013965
B	1.55220395788074	1.35748299844441	1.16499216883507	1.00109530216248

Example 2. Consider the following nonlinear fourth-order fractional integro-differential equation:

$$D^{\alpha}y(x) = 1 + \int_0^x e^{-t}y^2(t)\mathrm{d}t, \quad 0 < x < 1,\ 3 < \alpha \le 4 \tag{44}$$

subject to the following boundary conditions:

$$y(0) = 1, \qquad y''(0) = 1, \tag{45}$$

$$y(1) = e, \qquad y''(1) = e. \tag{46}$$

For $\alpha = 4$, the exact solution of the above problem (44)–(46) is

$$y(x) = e^x.$$

According to the variational iteration method, iteration formula (10) for Eq. (44) can be expressed in the following form:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6}J^{\alpha}\left(D^{\alpha}y_k(x) - 1 - \int_0^x e^{-t}y_k^2(t)\mathrm{d}t\right). \tag{47}$$

In order to avoid difficult fractional integration, we can take the truncated Taylor expansion for the exponential term in (47): e.g., $e^{-x} \sim 1 - x + x^2/2 - x^3/6$ and assume that an initial approximation has the following form which satisfies the initial conditions (45):

$$y_0(x) = 1 + Ax + \frac{x^2}{2} + \frac{B}{6}x^3 \tag{48}$$

where $A = y'(0)$ and $B = y'''(0)$ are unknowns to be determined.

Now, by iteration formula (47), the first-order approximation takes the following form:

$$\begin{aligned} y_1(x) &= y_0(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6}J^{\alpha}\left(D^{\alpha}y_k(x) - 1 - \int_0^x (1 - t + t^2/2 - t^3/6)y_k^2(t)\mathrm{d}t\right) \\ &= 1 + Ax + \frac{x^2}{2} + \frac{B}{6}x^3 - \frac{(\alpha-3)(\alpha-2)(\alpha-1)x^{\alpha}}{6} \\ &\quad \times \left(-\frac{1}{\Gamma(\alpha+1)} + x\left(-\frac{1}{\Gamma(\alpha+2)} + \frac{(1-2A)x}{\Gamma(\alpha+3)} + \cdots + \frac{1680B^2x^9}{\Gamma(\alpha+11)}\right)\right). \end{aligned} \tag{49}$$

By imposing boundary conditions (46) in $y_1(x)$, we obtain Table 3 which shows the values of A and B for different values of α.

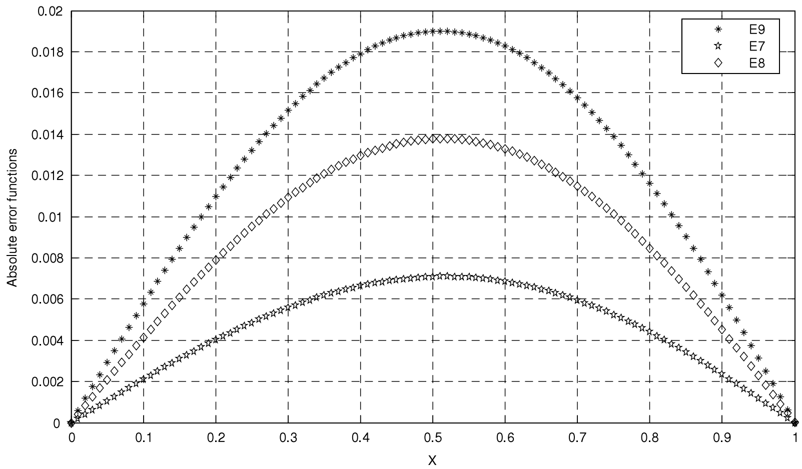

Fig. 5. Absolute error functions $E_7(x)$, $E_8(x)$ and $E_9(x)$ obtained by first-order VIM with $\alpha = 3.25$, $\alpha = 3.5$ and $\alpha = 3.75$.

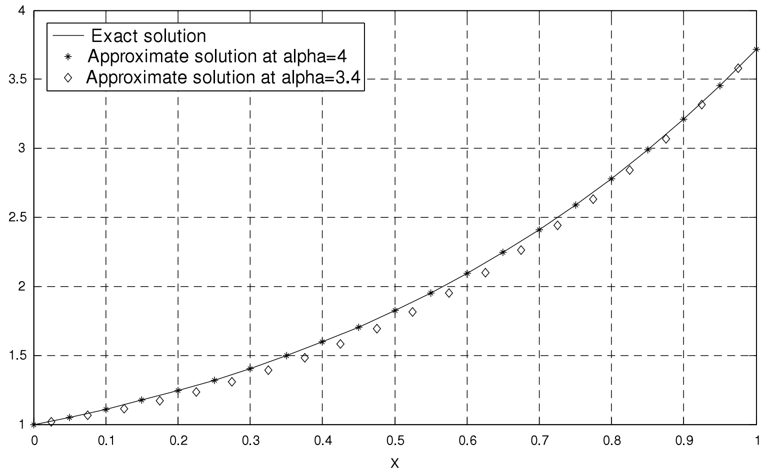

Fig. 6. Comparison of first-order approximate solution obtained by VIM with the exact solution at $\alpha = 4$ and $\alpha = 3.4$.

In Fig. 5, we draw absolute error functions $E_7(x) = |e^x - y_{1,3.75}|$, $E_8(x) = |e^x - y_{1,3.5}|$ and $E_9(x) = |e^x - y_{1,3.25}|$ for different values of α, where e^x is an exact solution of (44)–(46) and $y_{1,3.75}$, $y_{1,3.5}$ and $y_{1,3.25}$ represent the values of $y_1(x)$ at $\alpha = 3.75$, $\alpha = 3.5$ and $\alpha = 3.25$, respectively (see Fig. 6).

Now, we solve Eqs. (44)–(46) by homotopy perturbation method.

According to HPM, we construct the following homotopy:

$$D^\alpha y(x) = p\left(1 + \int_0^x e^{-t} y^2(t)\mathrm{d}t\right). \tag{50}$$

Substitution of (18) into (50) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0 : D^\alpha y_0 = 0, \tag{51}$$

$$p^1 : D^\alpha y_1 = 1 + \int_0^x e^{-t} y_0^2(t)\mathrm{d}t, \tag{52}$$

$$p^2 : D^\alpha y_2 = 2\int_0^x e^{-t} y_0(t) y_1(t)\mathrm{d}t, \tag{53}$$

$$p^3 : D^\alpha y_3 = \int_0^x e^{-t}(2y_0(t)y_2(t) + y_1^2(t))\mathrm{d}t, \tag{54}$$

$$\vdots$$

Table 4
Values of A and B for different values of α using (61).

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4$
A	1.00646865931986	1.01085715673040	1.00647005332874	0.99746675420551
B	0.34838722251386	0.59592879361901	0.59592879361901	1.01767767908914

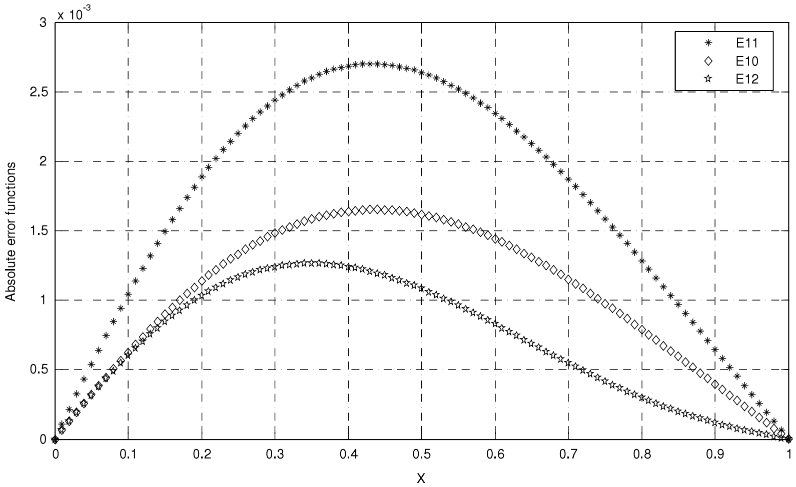

Fig. 7. Absolute error functions $E_{10}(x)$, $E_{11}(x)$ and $E_{12}(x)$ obtained by 2-term HPM with $\alpha = 3.25$, $\alpha = 3.5$ and $\alpha = 3.75$.

Applying the operator J^α to the above series of linear equations and using initial conditions (45), we get;

$$y_0(x) = 1, \tag{55}$$

$$y_1(x) = Ax + \frac{x^2}{2} + \frac{1}{6}Bx^3 + J^\alpha\left(1 + \int_0^x e^{-t}y_0^2(t)\mathrm{d}t\right), \tag{56}$$

$$y_{2n}(x) = J^\alpha\left(\int_0^x e^{-t}(2y_0(t)y_{2n-0}(t) + 2y_1(t)y_{2n-1}(t) + \cdots + 2y_{n-1}(t)y_{2n-(n-1)}(t) + y_n^2(t))\mathrm{d}t\right),$$
$$n = 1, 2, 3, 4, \ldots \tag{57}$$

$$y_{2n+1}(x) = J^\alpha\left(\int_0^x 2e^{-t}(y_0(t)y_{2n+1}(t) + y_1(t)y_{(2n+1)-1}(t) + \cdots + y_n(t)y_{2n+1-n}(t))\mathrm{d}t\right), \quad n = 1, 2, 3, 4, \ldots \tag{58}$$

where $A = y'(0)$ and $B = y'''(0)$ are to be determined.

In order to avoid difficult fractional integration, we can take the truncated Taylor expansions for the exponential term in (56)–(58): e.g., $e^{-x} \sim 1 - x + x^2/2 - x^3/6$.

Thus, by solving Eqs. (55)–(58), we obtain $y_1, y_2, \ldots$ e.g.:

$$y_1(x) = Ax + \frac{x^2}{2} + \frac{Bx^3}{6} + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{x^{\alpha+4}}{\Gamma(\alpha+5)} \tag{59}$$

$$y_2(x) = \frac{2Ax^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{4Ax^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{6x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{2Bx^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{12x^{\alpha+5}}{\Gamma(\alpha+6)} - \frac{8Ax^{\alpha+5}}{\Gamma(\alpha+6)} - \frac{6Bx^{\alpha+5}}{\Gamma(\alpha+6)}$$
$$- \frac{20x^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{20Bx^{\alpha+6}}{\Gamma(\alpha+7)} - \frac{40Bx^{\alpha+7}}{\Gamma(\alpha+8)} + \frac{2t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots. \tag{60}$$

Now, we can form the 2-term approximation

$$\phi_2(x) = 1 + Ax + \frac{x^2}{2} + \frac{Bx^3}{6} + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (2A-1)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{3x^{\alpha+3}}{\Gamma(\alpha+4)} - (2B+7)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)}$$
$$+ \frac{12x^{\alpha+5}}{\Gamma(\alpha+6)} - \frac{8Ax^{\alpha+5}}{\Gamma(\alpha+6)} - \frac{6Bx^{\alpha+5}}{\Gamma(\alpha+6)} - \frac{20x^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{20Bx^{\alpha+6}}{\Gamma(\alpha+7)} - \frac{40Bx^{\alpha+7}}{\Gamma(\alpha+8)} + \frac{2t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots \tag{61}$$

where A and B can be determined by imposing boundary conditions (46) on ϕ_2. Table 4 shows the values of A and B for different values of α. In Fig. 7, we draw absolute error functions, $E_{10}(x) = |e^x - \phi_{2,3.75}|$, $E_{11}(x) = |e^x - \phi_{2,3.5}|$ and

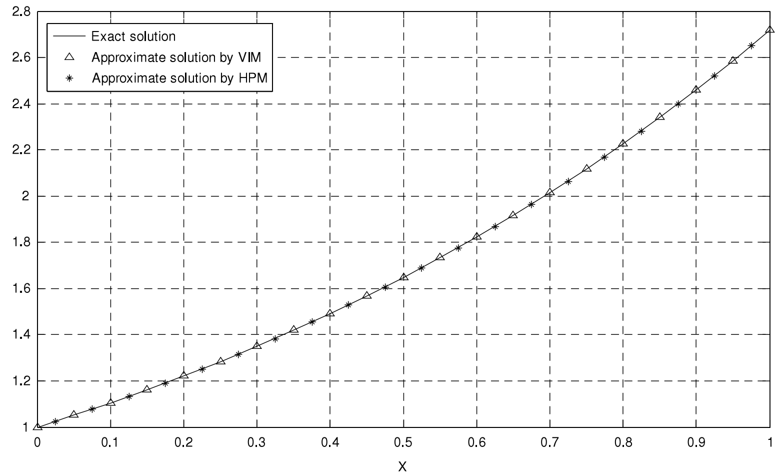

Fig. 8a. Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 4$.

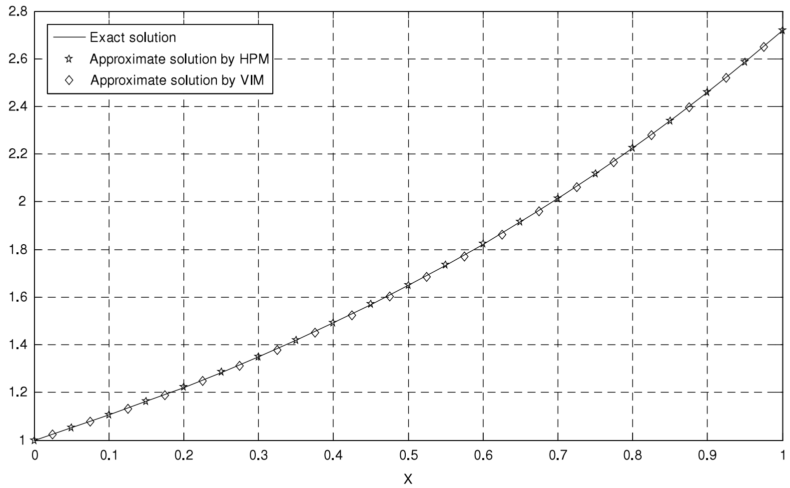

Fig. 8b. Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 3.8$.

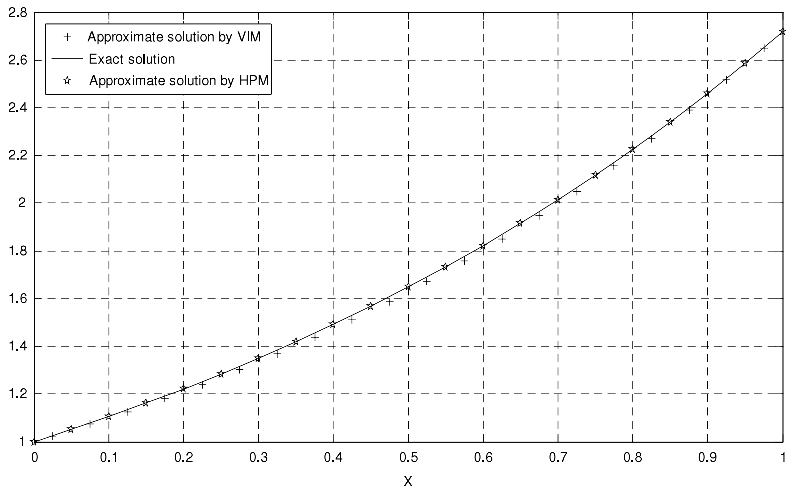

Fig. 8c. Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 3.2$.

$E_{12}(x) = |e^x - \phi_{2,3.25}|$ for different values of α, where e^x is an exact solution of (44)–(46) and $\phi_{2,3.75}$, $\phi_{2,3.5}$ and $\phi_{2,3.25}$ represent the values of ϕ_2 at $\alpha = 3.75$, $\alpha = 3.5$ and $\alpha = 3.25$, respectively. In Figs. 8a–8c, we compare the approximate solutions obtained by VIM and HPM with an exact solution, and it is clear from Figs. 8a–8c that the approximate solutions are in good agreement with an exact solution of (44)–(46) at $\alpha = 4$, $\alpha = 3.8$ and $\alpha = 3.2$. Also it is to be noted that the accuracy can be improved by computing more terms of approximated solutions and/or by taking more terms in the Taylor expansion for the exponential term.

6. Conclusion

In this article, variational iteration method (VIM) and homotopy perturbation method (HPM) have been successfully applied to linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations. Two examples are presented to illustrate the accuracy of the present schemes of VIM and HPM. Comparisons of VIM and HPM with exact solution have been shown by graphs and absolute error functions are plotted which show the efficiency of the methods.

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