The Landau criterion for Superfluidity

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A Bose-Einstein Condensate (BEC) is a medium constituted by identical bosonic particles at very low temperature that all share the same quantum wave function. Let's consider an impurity of mass *M,* moving inside a BEC, its interaction with the condensate being weak. At some point the impurity might create an excitation of energy $\hbar\,\omega_k$ and momentum $\hbar\,\vec{k}$. We assume that this excitation is well described by Bogoliubov's equations for small perturbations $\delta\varphi$ around the stationary solutions φ of the field equations for the system. In that case, the Landau criterion for superfluidity states that if the impurity velocity $\|\vec{v}\|$ is lower than a critical velocity v_c (equal to the BEC sound velocity), no excitation can be created (or destroyed) by the impurity. Otherwise, it would violate conservation of energy and momentum. So that, if $\|\vec{v}\| < v_c$ the impurity will move within the condensate without dissipation or momentum exchange, the condensate is superfluid (Phys. Rev. Lett. 85, 483 (2000)). Note: low temperature liquid ^{4}He is a well known example of superfluid that can, for instance, flow through narrow capillaries with no dissipation. However, for superfluid helium, the critical velocity is lower than the sound velocity. This is explained by the fact that liquid ^{4}He is a strongly interacting medium. We are here rather considering the case of weakly interacting cold atomic gases.

Landau criterion for superfluidity

Background: For a BEC close to its ground state (at temperature *T* = *0* K), its excitations are well described by small perturbations around the stationary state of the BEC. The energy of an excitation is then given by the Bogoliubov dispersion relation (derived previously in Mapleprimes "Quantum Mechanics using computer algebra II").

$$\varepsilon_k = \hbar\,\omega_k = \pm\sqrt{\frac{k^4\,\hbar^4}{4\,m^2} + \frac{k^2\,\hbar^2\,G\,n}{m}}\;;$$

where G is the atom-atom interaction constant, *n* is the density of particles, *m* is the mass of the condensed particles, *k* is the wave-vector of the excitations and ω_k their pulsation ($2\,\pi$ time the frequency). Typically, there are two possible types of excitations, depending on the wave-vector *k*:

- In the limit: $k \to 0$, $\varepsilon_k \sim \hbar\cdot k\cdot v_c$ with $v_c = \sqrt{\frac{G\,n}{m}}$, this relation is linear in *k* and is typical of a massless quasi-particle, i.e. a phonon excitation.
- In the limit: $k \to \infty$, $\varepsilon_k \sim \frac{\hbar^2\,k^2}{2\,m}$ which is the dispersion relation of a free particle of mass *m*, i.e. one single atom of the BEC.

Problem: An impurity of mass *M* moves with velocity $\vec{v}$ within such a condensate and creates an excitation with wave-vector $\vec{k}$. After the interaction process, the impurity is scattered with velocity $\vec{w}$.

a) Departing from Bogoliubov's dispersion relation, plus energy and momentum conservation, show that, in order to create an excitation, the impurity must move with an initial velocity

$$\|\vec{v}\| \geq v_c = \sqrt{\frac{G\,n}{m}}$$

When $\|\vec{v}\| < v_c$, no excitation can be created and the impurity moves through the medium without dissipation, as if the viscosity is 0, characterizing a superfluid. This is the Landau criterion for superfluidity.

b) Show that when the atom-atom interaction constant $G \geq 0$ (repulsive interactions), this value v_c is equal to the group velocity of the excitation (speed of sound in a condensate).

Solution

The version of Physics used below is from April/10 (or later), available at the Maplesoft Physics Research & Development webpage

a) The system consists of a condensate in its ground state and an impurity. Originally there is no excitation so the energy of the system can be approximated to the kinetic energy of the impurity. When an excitation with energy $\hbar\,\omega_k$ is created, the total energy will be the kinetic energy of the impurity after being scattered (final velocity is $\vec{w}$) plus the energy $\hbar\,\omega_k$ of the excitation.

Conservation of energy then implies on

> $restart : with(Physics) : with(Physics[Vectors]) : Setup(mathematicalnotation = true) :$

> $\frac{1}{2} \cdot M \cdot Norm(\vec{v})^2 = \frac{1}{2} \cdot M \cdot Norm(\vec{w})^2 + \hbar \cdot \omega_k$

$$\frac{M\,\|\vec{v}\|^2}{2} = \frac{M\,\|\vec{w}\|^2}{2} + \hbar\,\omega_k \quad \textbf{(1.1.1)}$$

Likewise, conservation of linear momentum implies on

> $M \cdot \vec{v} = M \cdot \vec{w} + \hbar \cdot \vec{k}$

$$M\,\vec{v} = \hbar\,\vec{k} + M\,\vec{w} \quad \textbf{(1.1.2)}$$

where $\vec{k}$ is the wave vector of the excitation with respect to the condensate (a referential with its origin at the condensates's center of mass). Before proceeding further, indicate the real variables of this problem

> $Setup\left(realobjects = \{\vec{k}, \vec{v}, \hbar, m, M, G\}\right) :$

The key observation now is that, for an excitation in a state that is a small perturbation around a stationary state, the wave vector $\vec{k}$ of the excitation is related to its energy by Bogoliubov's dispersion relation

> $\hbar\,\omega_k = \sqrt{\frac{k^4\,\hbar^4}{4\,m^2} + \frac{k^2\,\hbar^2\,G\,n}{m}}$

$$\hbar\,\omega_k = \frac{\sqrt{\frac{k^4\,\hbar^4}{m^2} + \frac{4\,k^2\,\hbar^2\,G\,n}{m}}}{2} \qquad \textbf{(1.1.3)}$$

where $k = \left\|\vec{k}\right\|$. So to obtain the minimum (critical) value of $\|\vec{v}\|$ such that an excitation exists (is created) we need to combine these three equations **(1.1.1)**, **(1.1.2)** and **(1.1.3)**.

Start removing $\vec{w}$ from **(1.1.1)** using **(1.1.2)**

> *isolate*(**(1.1.2)**, $w\_$)

$$\vec{w} = -\frac{\hbar\,\vec{k} - M\,\vec{v}}{M} \qquad \textbf{(1.1.4)}$$

> *expand*(*subs*(**(1.1.4)**, **(1.1.1)**))

$$\frac{M\,\|\vec{v}\|^2}{2} = \frac{\hbar^2\,\left\|\vec{k}\right\|^2 - 2\,M\,\hbar\,\left(\vec{k}\cdot\vec{v}\right) + M^2\,\|\vec{v}\|^2}{2\,M} + \hbar\,\omega_k \qquad \textbf{(1.1.5)}$$

Eliminate now ω_k using **(1.1.3)**. On the way, rewrite $\vec{k}\cdot\vec{v}$ as $\left\|\vec{k}\right\|\,\|\vec{v}\|\cdot\cos(\theta)$, where θ is the angle between the two vectors

> *subs*$\left(\left\{\textbf{(1.1.3)}, \vec{k}\cdot\vec{v} = k\cdot\|\vec{v}\|\cdot\cos(\theta), \left\|\vec{k}\right\| = k\right\}, \textbf{(1.1.5)}\right)$

$$\frac{M\,\|\vec{v}\|^2}{2} = \frac{\frac{k^2\,\hbar^2}{2} - M\,\hbar\,k\,\|\vec{v}\|\,\cos(\theta) + \frac{M^2\,\|\vec{v}\|^2}{2}}{M} + \frac{\sqrt{\frac{k^4\,\hbar^4}{m^2} + \frac{4\,k^2\,\hbar^2\,G\,n}{m}}}{2} \qquad \textbf{(1.1.6)}$$

Isolate now the norm of $\vec{v}$

> *isolate*(**(1.1.6)**, *Norm*($v\_$))

$$\|\vec{v}\| = -\frac{-\frac{\sqrt{\frac{k^4\,\hbar^4}{m^2} + \frac{4\,k^2\,\hbar^2\,G\,n}{m}}\,M}{\hbar\,k} - \hbar\,k}{2\,M\cos(\theta)} \qquad \textbf{(1.1.7)}$$

Simplify this result taking into account that $G \geq 0$, $m > 0$, $k > 0$, $\hbar > 0$

> *simplify*(**(1.1.7)**) assuming $G \geq 0, m > 0, k > 0, \hbar > 0$

$$\|\vec{v}\| = \frac{\hbar\,k\,m + \sqrt{k^2\,\hbar^2 + 4\,G\,m\,n}\;M}{2\,m\,M\cos(\theta)} \qquad \textbf{(1.1.8)}$$

Since $k \equiv \left\|\vec{k}\right\| \geq 0$, in the case $G \geq 0$, the minimum possible value for $\|\vec{v}\|$ (which must be kept positive , i.e. $\cos(\theta) > 0$) is obtained for the lowest k values, i.e. in the phonon regime for the excitations, as discussed in the **Background:**

> $\lim_{k \to 0}$ **(1.1.8)**

$$\|\vec{v}\| = \frac{\sqrt{G\,m\,n}}{m\cos(\theta)} \qquad \textbf{(1.1.9)}$$

Combine the square roots

> *combine*(*simplify*(**(1.1.9)**)) assuming $m > 0, n > 0, G \geq 0$

$$\|\vec{v}\| = \frac{\sqrt{\frac{G\,n}{m}}}{\cos(\theta)} \qquad \textbf{(1.1.10)}$$

Finally, the lowest possible value for $\|\vec{v}\|$, the critical velocity v_c, is obtained minimizing with respect to θ, that is, for $\cos(\theta) = 1$:

> $v_c = minimize\left(rhs(\textbf{(1.1.10)}), \theta = -\frac{\pi}{2} .. \frac{\pi}{2}\right)$ assuming $m > 0, n > 0, G \geq 0$

$$v_c = \sqrt{\frac{G\,n}{m}} \qquad \textbf{(1.1.11)}$$

In summary: the three equations **(1.1.1)**, **(1.1.2)** and **(1.1.3)** together, that is conservation of energy, momentum and Bogoliubov's dispersion relation for an excitation of the condensate, imply that, for an excitation to be scattered by the impurity, $\|\vec{v}\| \geq v_c = \sqrt{\frac{G\,n}{m}}$. Below v_c there is no solution for $\|\vec{v}\|$, so no interaction, in which case the motion is superfluid. One could also notice that if $G = 0$, v_c=0, that is, the superfluidity behavior is closely related to the existence of an atom-atom interaction term, represented by the non-linear term of the Gross-Pitaevskii equation (derived previously in Mapleprimes "Quantum Mechanics using Computer Algebra"). Note: Bogoliubov spectrum has two branches. In **(1.1.3)**, we have chosen the '+' one, but the conclusions are the same for the other one.

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b) The group velocity is given by $v_g = \frac{\partial \omega_k}{\partial k}$, so departing from Bogoliubov's dispersion relation **(1.1.3)** we have

> $\frac{\textbf{(1.1.3)}}{\hbar}$

$$\omega_k = \frac{\sqrt{\frac{k^4\,\hbar^4}{m^2} + \frac{4\,k^2\,\hbar^2\,G\,n}{m}}}{2\,\hbar} \qquad \textbf{(1.1.12)}$$

> $v_g = diff(rhs(\textbf{(1.1.12)}), k)$

(1.1.13)

$$v_g = \frac{\frac{4\,k^3\,\hbar^4}{m^2} + \frac{8\,k\,\hbar^2\,G\,n}{m}}{4\,\sqrt{\frac{k^4\,\hbar^4}{m^2} + \frac{4\,k^2\,\hbar^2\,G\,n}{m}}\;\hbar} \tag{1.1.13}$$

> *simplify*((**1.1.13**)) assuming $k > 0, m > 0, \hbar > 0$

$$v_g = \frac{k^2\,\hbar^2 + 2\,G\,m\,n}{m\,\sqrt{k^2\,\hbar^2 + 4\,G\,m\,n}} \tag{1.1.14}$$

For $G \geq 0$, and in the limit $k \rightarrow 0$ (phonon branch), considered in the previous section (see (**1.1.9**)) we have

> *limit*((**1.1.14**), $k = 0$)

$$v_g = \frac{G\,n}{\sqrt{G\,m\,n}} \tag{1.1.15}$$

Combining the square roots

> *combine*(*simplify*((**1.1.15**))) assuming $G \geq 0, n > 0, m > 0$

$$v_g = \sqrt{\frac{G\,n}{m}} \tag{1.1.16}$$

In summary, the group velocity v_g (speed of sound in a condensate) is equal to the critical velocity v_c that an impurity needs to have in order to move within the condensate creating an excitation and dissipation.

References

[1] Suppression and enhancement of impurity scattering in a Bose-Einstein condensate
[2] Superfluidity versus Bose-Einstein condensation
[3] Bose–Einstein condensate (wiki)
[4] Dispersion relations (wiki)