Why computer algebra?

... and why computers?

We can concentrate more on the ideas instead of on the algebraic manipulations

We can extend results with ease

We can explore the mathematics surrounding a problem

We can share results in a reproducible way

Representation issues that were preventing the use of computer algebra in Physics

Notation and related mathematical methods that were missing:

coordinate free representations for vectors and vectorial differential operators,
covariant tensors distinguished from contravariant tensors,
functional differentiation, relativity differential operators and sum

rule for tensor contracted (repeated) indices
Bras, Kets, projectors and all related to Dirac's notation in Quantum Mechanics

Inert representations of operations, mathematical functions, and related typesetting were missing:

inert versus active representations for mathematical operations
ability to move from inert to active representations of computations and viceversa as necessary
hand-like style for entering computations and textbook-like notation for displaying results

Key elements of the computational domain of theoretical physics were missing:

ability to handle products and derivatives involving commutative, anticommutative and noncommutative variables and functions
ability to perform computations taking into account custom-defined algebra rules of different kinds
(commutator, anticommutator and bracket rules, etc.)

Examples

The Maple computer algebra environment

- In the presentation that follows we use the Maple worksheet mode, where input lines are identified by a prompt, as in:

```
>
```

- We communicate with the computer entering our computation typing in this input line. The output is the result of our computation and automatically gets an equation number than we can later refer to

```
> (%int = int)(cos(x), x)
```

$$\int \cos(x)\, dx = \sin(x) \quad (3.1)$$

• To refer to an equation, you enter the equation label by pressing Command + L, then typing the equation number as you see it

> **(3.1)**

$$\int \cos(x)\, dx = \sin(x) \quad (3.2)$$

>

Classical Mechanics

Inertia tensor for a triatomic molecule

Problem

Determine the Inertia tensor of a triatomic molecule that has the form of an isosceles triangle with two masses m_1 in the extremes of the base and mass m_2 in the third vertex. The distance between the two masses m_1 is equal to *a,* and the height of the triangle is equal to *h*.

Solution

> *restart*; *with*(*Physics*, *KroneckerDelta*) : *with*(*Physics*[*Vectors*]) :

The general formula

> *InertiaTensor* := *Sum*(*m*[*k*] (*Norm*(*r_*[*k*])^2 *KroneckerDelta*[*i*, *j*] - *Component*(*r_*[*k*], *i*) *Component*(*r_*[*k*], *j*)), *k* = 1 .. *N*)

$$InertiaTensor := \sum_{k=1}^{N} m_k \left(\left\| \vec{r}_k \right\|^2 \delta_{i,j} - \left(\vec{r}_k\right)_i \left(\vec{r}_k\right)_j \right) \quad (4.1.1.1)$$

There are 3 particles

> *N* := 3

$$N := 3 \quad (4.1.1.2)$$

Create an indexing function

> *IT* := *unapply*(*InertiaTensor*, *i*, *j*)

$$IT := (i,j) \mapsto \sum_{k=1}^{3} m_k \left(\left\| \vec{r}_k \right\|^2 \delta_{i,j} - \left(\vec{r}_k\right)_i \left(\vec{r}_k\right)_j \right) \quad (4.1.1.3)$$

The inerta tensor matrix

> *IT_Matrix* := *Matrix*(3, *IT*)

$$IT\_Matrix := \left[\left[\sum_{k=1}^{3} m_k \left(\left\| \vec{r}_k \right\|^2 - \left(\vec{r}_k\right)_1^{\,2} \right), \sum_{k=1}^{3} \left(-m_k \left(\vec{r}_k\right)_1 \left(\vec{r}_k\right)_2 \right), \sum_{k=1}^{3} \left(\right. \right. \right. \quad (4.1.1.4)$$

$$-m_k \left(\vec{r}_k\right)_1 \left(\vec{r}_k\right)_3\Big)\Big],$$

$$\left[\sum_{k=1}^{3}\left(-m_k \left(\vec{r}_k\right)_1 \left(\vec{r}_k\right)_2\right), \sum_{k=1}^{3} m_k \left(\left\|\vec{r}_k\right\|^2 - \left(\vec{r}_k\right)_2^{\,2}\right), \sum_{k=1}^{3}\left(-m_k \left(\vec{r}_k\right)_2 \left(\vec{r}_k\right)_3\right)\right],$$

$$\left[\sum_{k=1}^{3}\left(-m_k \left(\vec{r}_k\right)_1 \left(\vec{r}_k\right)_3\right), \sum_{k=1}^{3}\left(-m_k \left(\vec{r}_k\right)_2 \left(\vec{r}_k\right)_3\right), \sum_{k=1}^{3} m_k \left(\left\|\vec{r}_k\right\|^2 - \left(\vec{r}_k\right)_3^{\,2}\right)\right]\Big]\Big]$$

Choose a system of reference (not at the center of mass). The vectors $\vec{r}_k$ are related to $\vec{R}_k$ and $\vec{R}_{CM}$ by

> *position* := r_[k] = R_[k]-R_[CM];

$$position := \vec{r}_k = \vec{R}_k - \vec{R}_{CM} \qquad \textbf{(4.1.1.5)}$$

Choose the origin at the middle of the segment connecting the two atoms of equal mass

> R_[1] := - $\frac{a}{2}$ _i;

$$\vec{R}_1 := -\frac{a\,\hat{i}}{2} \qquad \textbf{(4.1.1.6)}$$

> R_[2] := h _k

$$\vec{R}_2 := h\,\hat{k} \qquad \textbf{(4.1.1.7)}$$

> R_[3] := $\frac{a}{2}$ _i

$$\vec{R}_3 := \frac{a\,\hat{i}}{2} \qquad \textbf{(4.1.1.8)}$$

Two masses are equal

> m_3 := m[1]

$$m_3 := m_1 \qquad \textbf{(4.1.1.9)}$$

The "*center of mass*"

> R_[CM] := Sum(m[j] R_[j], j = 1 .. N) / Sum(m[j], j = 1 .. N);

$$\vec{R}_{CM} := \frac{\sum_{j=1}^{3} m_j \vec{R}_j}{\sum_{j=1}^{3} m_j} \qquad \textbf{(4.1.1.10)}$$

> $\vec{R}_{CM} := value\left(\vec{R}_{CM}\right)$

$$\vec{R}_{CM} := \frac{m_2\, h\, \hat{k}}{2\, m_1 + m_2} \tag{4.1.1.11}$$

The positions of the three particles viewed from the center of mass

> *position*

$$\vec{r}_k = \vec{R}_k - \frac{m_2\, h\, \hat{k}}{2\, m_1 + m_2} \tag{4.1.1.12}$$

> $seq(position, k = 1 ..N)$

$$\vec{r}_1 = -\frac{a\, \hat{i}}{2} - \frac{m_2\, h\, \hat{k}}{2\, m_1 + m_2}, \vec{r}_2 = h\, \hat{k} - \frac{m_2\, h\, \hat{k}}{2\, m_1 + m_2}, \vec{r}_3 = \frac{a\, \hat{i}}{2} - \frac{m_2\, h\, \hat{k}}{2\, m_1 + m_2} \tag{4.1.1.13}$$

The abstract IT_Matrix at these values of the vectors $\vec{r}_k$

> $IT\_answer := simplify(eval(value(IT\_Matrix), [\mathbf{(4.1.1.13)}]))$

$$IT\_answer := \begin{bmatrix} \frac{2\, m_2\, h^2\, m_1}{2\, m_1 + m_2} & 0 & 0 \\ 0 & \frac{2\, a^2\, m_1^2 + m_2\, (a^2 + 4\, h^2)\, m_1}{4\, m_1 + 2\, m_2} & 0 \\ 0 & 0 & \frac{m_1\, a^2}{2} \end{bmatrix} \tag{4.1.1.14}$$

>

Try changing the value of m_3 in **(4.1.1.9)** and re-execute the lines after that definition and you see the corresponding answer instantly - computer algebra allows for these simple recalculations without having to reformulate anything.

Classical Field Theory

The field equations for the $\lambda\, \Phi^4$ model

The Lagrangian of the $\lambda\, \Phi^4$ model, the corresponding Action, and the field equations:

> *restart*; *with*(*Physics*) :

> *Coordinates*($X = cartesian$)

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (x, y, z, t)\}$

Systems of spacetime Coordinates are: $\{X = (x, y, z, t)\}$

$$\{X\} \tag{5.1.1}$$

> *CompactDisplay*($\Phi(X)$)

$$\Phi(x, y, z, t) \textit{ will now be displayed as } \Phi \tag{5.1.2}$$

> $L := \frac{1}{2}\, d_{\mu}(\Phi(X))^2 - \frac{m^2}{2}\,\Phi(X)^2 + \left(\frac{\lambda}{4}\,\Phi(X)^4\right)$

$$L := \frac{\left(\partial_{\mu}(\Phi)\right)\left(\partial^{\mu}(\Phi)\right)}{2} - \frac{m^2\,\Phi^2}{2} + \frac{\lambda\,\Phi^4}{4} \qquad \textbf{(5.1.3)}$$

> $S := Intc(L, X)$

$$S := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\left(\partial_{\mu}(\Phi)\right)\left(\partial^{\mu}(\Phi)\right)}{2} - \frac{m^2\,\Phi^2}{2} + \frac{\lambda\,\Phi^4}{4}\right) dx\, dy\, dz\, dt \qquad \textbf{(5.1.4)}$$

> $(\%Fundiff = Fundiff)(S, \Phi)$

$$\left(\frac{\delta}{\delta\,\Phi}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\left(\partial_{\mu}(\Phi)\right)\left(\partial^{\mu}(\Phi)\right)}{2} - \frac{m^2\,\Phi^2}{2} + \frac{\lambda\,\Phi^4}{4}\right) dx\, dy\, dz\, dt = -\square(\Phi) - \Phi\left(-\Phi^2\,\lambda + m^2\right) \qquad \textbf{(5.1.5)}$$

> $convert(\%, diff)$

$$\left(\frac{\delta}{\delta\,\Phi}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\left(\partial_{\mu}(\Phi)\right)\left(\partial^{\mu}(\Phi)\right)}{2} - \frac{m^2\,\Phi^2}{2} + \frac{\lambda\,\Phi^4}{4}\right) dx\, dy\, dz\, dt = \Phi_{x,x} + \Phi_{y,y} + \Phi_{z,z} - \Phi_{t,t} - \Phi\left(-\Phi^2\,\lambda + m^2\right) \qquad \textbf{(5.1.6)}$$

Maxwell equations departing from the 4-dimensional Action for Electrodynamics

Maxwell equations result from equating to zero the functional derivative o the Action with respect to the 4-D potential A_{μ}

> $restart;\ with(Physics):$

> $Coordinates(X = Cartesian)$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (x, y, z, t)\}$

Systems of spacetime Coordinates are: $\{X = (x, y, z, t)\}$

$$\{X\} \qquad \textbf{(5.2.1)}$$

The 4-D electromagnetic potential

> $Define(A[\mathrm{mu}](X))$

Defined objects with tensor properties

$$\left\{A_{\mu}, \gamma_{\mu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \qquad \textbf{(5.2.2)}$$

> $CompactDisplay(A(X))$

$$A(x, y, z, t)\ \textit{will now be displayed as}\ A \qquad \textbf{(5.2.3)}$$

The electromagnetic field tensor $F_{\mu,\nu}$

> $F[\mathrm{mu}, \mathrm{nu}] := d\_[\mathrm{mu}](A[\mathrm{nu}](X)) - d\_[\mathrm{nu}](A[\mathrm{mu}](X));$

(5.2.4)

$$F_{\mu,\nu} := \partial_\mu(A_\nu) - (\partial_\nu(A_\mu)) \quad \textbf{(5.2.4)}$$

The functional derivative of the corresponding Action

> 'Fundiff'(Intc(F[mu, nu]^2, X), A[rho]) = 0

$$\left(\frac{\delta}{\delta A_\rho}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\partial_\mu(A_\nu) - (\partial_\nu(A_\mu))\right)^2 \,dx\,dy\,dz\,dt = 0 \quad \textbf{(5.2.5)}$$

> **(5.2.5)**

$$\left(2\left(\partial_\mu\left(\partial_\nu\left(A^\nu\right)\right)\right) - 2\,\Box\left(A_\mu\right)\right)g^{\mu,\rho} + \left(-2\,\Box\left(A_\nu\right) + 2\left(\partial_\mu\left(\partial_\nu\left(A^\mu\right)\right)\right)\right)g^{\nu,\rho} = 0 \quad \textbf{(5.2.6)}$$

Simplify the contracted spacetime indices

> *Simplify*(**(5.2.6)**)

$$-4\,\Box\left(A^\rho\right) + 4\left(\partial^\mu\left(\partial^\rho\left(A_\mu\right)\right)\right) = 0 \quad \textbf{(5.2.7)}$$

The system of differential equations behind this formula in standard Maple notation:

> *Library:-ToContravariant*(**(5.2.7)**)

$$4\,g_{\mu,\nu}\left(\partial^\mu\left(\partial^\rho\left(A^\nu\right)\right)\right) - 4\,\Box\left(A^\rho\right) = 0 \quad \textbf{(5.2.8)}$$

> *convert*(*Library:-TensorComponents*(**(5.2.8)**), *diff*)

$$\Big[-4\left(A^2\right)_{x,y} - 4\left(A^3\right)_{x,z} - 4\left(A^4\right)_{t,x} + 4\left(A^1\right)_{y,y} + 4\left(A^1\right)_{z,z} - 4\left(A^1\right)_{t,t} = 0, \quad \textbf{(5.2.9)}$$

$$-4\left(A^1\right)_{x,y} - 4\left(A^3\right)_{y,z} - 4\left(A^4\right)_{t,y} + 4\left(A^2\right)_{x,x} + 4\left(A^2\right)_{z,z} - 4\left(A^2\right)_{t,t} = 0,$$

$$-4\left(A^1\right)_{x,z} - 4\left(A^2\right)_{y,z} - 4\left(A^4\right)_{t,z} + 4\left(A^3\right)_{x,x} + 4\left(A^3\right)_{y,y} - 4\left(A^3\right)_{t,t} = 0,$$

$$4\left(A^1\right)_{t,x} + 4\left(A^2\right)_{t,y} + 4\left(A^3\right)_{t,z} + 4\left(A^4\right)_{x,x} + 4\left(A^4\right)_{y,y} + 4\left(A^4\right)_{z,z} = 0\Big]$$

>

The Gross-Pitaevskii field equations for a quantum system of identical particles

Problem: derive the field equation describing the ground state of a quantum system of identical particles (bosons), that is, the Gross-Pitaevskii equation (GPE). This equation is useful to describe Bose-Einstein condensates (BEC).

Solution

Two steps:

- Construct the Lagrangian for the system, and with it write the action functional
- Minimize the action by equating to zero its functional derivative with respect to the boson field.

```
> restart; with(Physics) : with(Physics[Vectors]) :
> interface(imaginaryunit = i) :
> macro(Psi = psi(x, y, z, t)) :
> CompactDisplay((psi, V)(x, y, z, t))
```

$\psi(x, y, z, t)$ *will now be displayed as* ψ

$V(x, y, z, t)$ *will now be displayed as* V **(5.3.1.1)**

The energy density E for a quantum system of identical boson particles is (see [3])

```
> E := hbar^2/(2 m) Norm(%Gradient(Psi))^2 + V(x, y, z, t) abs(Psi)^2 + G/2 abs(Psi)^4;
```

$$E := \frac{\hbar^2 \left\|\nabla\psi\right\|^2}{2\,m} + V\left|\psi\right|^2 + \frac{G\left|\psi\right|^4}{2} \tag{5.3.1.2}$$

$\psi(x, y, z, t)$ is a complex field, $V(x, y, z, t)$ an external potential, the term $\frac{G\left|\psi\right|^4}{2}$ is the atom-atom interaction.

```
> Setup(realobjects = {t, m, hbar, G, V(x, y, z, t)}, automaticsimplification = true) :
```

The Lagrangian density L in terms of the Energy E

```
> L := ((i hbar)/2) (conjugate(Psi) diff(Psi, t) - Psi diff(conjugate(Psi), t)) - E
```

$$L := \frac{-\mathrm{i}\,\hbar\,\psi\,\overline{\psi}_t\,m - \hbar^2\left\|\nabla\psi\right\|^2 + \left(-G\left|\psi\right|^4 + \mathrm{i}\,\psi_t\,\overline{\psi}\,\hbar - 2\,V\left|\psi\right|^2\right)m}{2\,m} \tag{5.3.1.3}$$

Construct the action and equate to zero the functional derivative

```
> 'Fundiff'(Intc(L, x, y, z, t), psi) = 0
```

$$\left(\frac{\delta}{\delta\psi}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-\mathrm{i}\,\hbar\,\psi\,\overline{\psi}_t\,m - \hbar^2\left\|\nabla\psi\right\|^2 + \left(-G\left|\psi\right|^4 + \mathrm{i}\,\psi_t\,\overline{\psi}\,\hbar - 2\,V\left|\psi\right|^2\right)m}{2\,m}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t = 0 \tag{5.3.1.4}$$

```
> (5.3.1.4)
```

$$\frac{-2\,G\,\psi\,\overline{\psi}^2\,m - 2\,\mathrm{i}\,\hbar\,\overline{\psi}_t\,m + \hbar^2\,\overline{\psi}_{x,x} + \overline{\psi}_{y,y}\,\hbar^2 + \hbar^2\,\overline{\psi}_{z,z} - 2\,\overline{\psi}\,V\,m}{2\,m} = 0 \tag{5.3.1.5}$$

Make the Laplacian explicit

```
> (Laplacian = %Laplacian)(Psi)
```

$$\psi_{x,x} + \psi_{y,y} + \psi_{z,z} = \nabla^2\psi \tag{5.3.1.6}$$

```
> simplify(conjugate((5.3.1.5)), {(5.3.1.6)})
```

$$\frac{2 \mathrm{i} \hbar \psi_t m + \hbar^2 \nabla^2 \psi - 2 m \psi \left(G \overline{\psi} \psi + V\right)}{2 m} = 0 \qquad (5.3.1.7)$$

The standard form of the Gross–Pitaevskii equation:

> $i\,\hbar$ *isolate*((**5.3.1.7**), *diff*(Psi, t))

$$\mathrm{i} \hbar \psi_t = \frac{-\hbar^2 \nabla^2 \psi + 2 m \psi \left(G \overline{\psi} \psi + V\right)}{2 m} \qquad (5.3.1.8)$$

> *collect*(*convert*(*expand*((**5.3.1.8**)), abs), psi)

$$\mathrm{i} \hbar \psi_t = \left(G |\psi|^2 + V\right) \psi - \frac{\hbar^2 \nabla^2 \psi}{2 m} \qquad (5.3.1.9)$$

- For a continuation of this computation deriving a continuity equation for a system of identical particles, see the Mapleprimes post "Quantum Mechanics using Computer Algebra" .
- For the Bogoliubov spectrum and dispersion relations of this problem above see the Mapleprimes post "Quantum Mechanics II".
- For a derivation of the Landau criterion for superfluidity in a system of identical particles see the Mapleprimes post "Superfluidity in Quantum Mechanics"

>

Quantum mechanics

*The quantum operator components of $\vec{L}$ satisfy $[L_j, L_k]_- = i\,\epsilon_{j,k,m} L_m$

> *restart*; *with*(*Physics*) : *interface*(*imaginaryunit* = i) :

> *Setup*(*spaceindices* = *lowercaselatin*, *metric* = *Euclidean*, *automaticsimplification* = *true*)

The Euclidean metric in cartesian coordinates

Changing the signature of the tensor spacetime to: + + + +

[*automaticsimplification* = *true*, *metric* = {(1, 1) = 1, (2, 2) = 1, (3, 3) = 1, (4, 4) = 1}, *spaceindices* = *lowercaselatin*] **(6.1.1)**

Define L, r and p as tensors of the 3-D Euclidean space embedded in

> *Define*(L, r, p)

Defined objects with tensor properties

$$\left\{L, p, r, \gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \qquad (6.1.2)$$

Now set the related **Commutator** rules for the algebra in tensor notation

> *Setup*(*quantumoperators* = {L, p, r},
{%*Commutator*(p[j], p[k]) = 0,
%*Commutator*(r[j], p[k]) = i *KroneckerDelta*[j, k],
%*Commutator*(r[j], r[k]) = 0})

(6.1.3)

$$\left[algebrarules = \left\{ [p_j, p_k]_- = 0, [r_j, p_k]_- = \mathrm{i}\, \delta_{j,k}, [r_j, r_k]_- = 0 \right\}, quantumoperators = \{L, p, r\}\right] \quad (6.1.3)$$

Verify how these algebra rules work:

> (%Commutator = Commutator)(r[j], p[k])

$$[r_j, p_k]_- = \mathrm{i}\, \delta_{j,k} \quad (6.1.4)$$

> (%Commutator = Commutator)(r[j], r[k])

$$[r_j, r_k]_- = 0 \quad (6.1.5)$$

> (%Commutator = Commutator)(p[j], p[k]);

$$[p_j, p_k]_- = 0 \quad (6.1.6)$$

The definition of L_j

> L[j] = LeviCivita[j, k, m] r[k] p[m]

$$L_j = \epsilon_{j,k,m} r_k p_m \quad (6.1.7)$$

The rule to be verified

> %Commutator(L[j], L[k]) = i LeviCivita[j, k, m] L[m]

$$[L_j, L_k]_- = \mathrm{i}\, \epsilon_{j,k,m} L_m \quad (6.1.8)$$

Substitute now the operator L_i by its tensor form in terms r_k and p_m in the commutator above

> Library:-SubstituteTensor((6.1.7), (6.1.8))

$$[\epsilon_{j,a,m} r_a p_m, \epsilon_{k,b,c} r_b p_c]_- = \mathrm{i}\, \epsilon_{m,a,b} r_a p_b \epsilon_{j,k,m} \quad (6.1.9)$$

Simplify all in one go:

> Simplify((6.1.9))

$$\mathrm{i}\, (-r_k p_j + r_j p_k) = \mathrm{i}\, (-r_k p_j + r_j p_k) \quad (6.1.10)$$

The same one step at a time,

> expand((6.1.9))

$$\epsilon_{a,j,m} \epsilon_{b,c,k} (r_b p_c r_a p_m - r_a p_m r_b p_c) = \mathrm{i}\, \epsilon_{a,b,m} \epsilon_{j,k,m} r_a p_b \quad (6.1.11)$$

> Simplify((6.1.11))

$$\mathrm{i}\, (-r_k p_j + r_j p_k) = \mathrm{i}\, (-r_k p_j + r_j p_k) \quad (6.1.12)$$

>

Quantization of the energy of a particle in a magnetic field

Show that the energy of a particle in a constant magnetic field oriented along the z axis can be written as

$$H = \hbar\, \omega_c \left(a^\dagger a + \frac{1}{2} \right)$$

where $a^\dagger$ and a are creation and anihilation operators.

>

Solution

The classical Hamiltonian is given by

$$H=\frac{\left(\vec{p}-\frac{q\,\vec{A}}{c}\right)^2}{2\,m}$$

The underlying quantum mechanics algebra rules are

$$\left[(\vec{r})_i,(\vec{p})_j\right]_-=\delta_{i,j},\quad \left[(\vec{r})_i,(\vec{r})_j\right]_-=0,\quad \left[(\vec{p})_i,(\vec{p})_j\right]_-=0$$

> $restart; with(Physics) : with(Vectors) : interface(imaginaryunit = i) :$

> $Setup\left(hermitianoperators=\{\vec{A},H,\Pi,\vec{\Pi},p,\vec{p},x,y,z\},\ quantumoperators=\{a\},\ realobjects=\{\hbar,B,c,k,m,q,\omega_c\}\right);$

$$\left[hermitianoperators=\{\vec{A},H,\Pi,\vec{\Pi},p,\vec{p},x,y,z\},\ quantumoperators=\{\vec{A},H,\Pi,\vec{\Pi},a,p,\vec{p},x,y,z\},\ realobjects=\{\hbar,B,\hat{i},\hat{j},\hat{k},\hat{\phi},\hat{r},\hat{\rho},\hat{\theta},c,k,m,\phi,q,r,\rho,\theta,x,y,z,\omega_c\}\right] \quad (6.2.1.1)$$

The Hamiltonian

> $H=\frac{\vec{\Pi}^2}{2\ m}$

$$H=\frac{\vec{\Pi}^2}{2\,m} \quad (6.2.1.2)$$

where

> $\vec{\Pi}=p\_-\frac{q}{c}\cdot A\_(x,y)$

$$\vec{\Pi}=\vec{p}-\frac{q\,\vec{A}(x,y)}{c} \quad (6.2.1.3)$$

> $CompactDisplay(A\_(x,y))$

$$\vec{A}(x,y)\ will\ now\ be\ displayed\ as\ \vec{A} \quad (6.2.1.4)$$

> $Setup\left(\left\{[x,p_x]_-=i\,\hbar,\ [x,p_y]_-=0,\ [y,x]_-=0,\ [y,p_x]_-=0,\ [y,p_y]_-=i\,\hbar,\ [p_y,p_x]_-=0\right\}\right)$

$$\left[algebrarules=\left\{[x,p_x]_-=\mathrm{i}\,\hbar,\ [x,p_y]_-=0,\ [y,x]_-=0,\ [y,p_x]_-=0,\ [y,p_y]_-=\mathrm{i}\,\hbar,\ [p_y,p_x]_-=0\right\}\right] \quad (6.2.1.5)$$

In Coulomb's gauge, the following vector potential gives the magnetic field of the problem,

$\vec{B} = B\,\hat{k}$

> $A\_(x, y) = -\frac{B \cdot y}{2} \cdot \_i + \frac{B \cdot x}{2} \cdot \_j;$

$$\vec{A} = -\frac{1}{2}\,B\,\hat{i}\,y + \frac{1}{2}\,B\,\hat{j}\,x \quad \textbf{(6.2.1.6)}$$

Indeed we have

> $Divergence(\textbf{(6.2.1.6)})$

$$\nabla \cdot \vec{A} = 0 \quad \textbf{(6.2.1.7)}$$

> $Curl(\textbf{(6.2.1.6)})$

$$\nabla \times \vec{A} = B\,\hat{k} \quad \textbf{(6.2.1.8)}$$

Derive now the commutation rule for $[\Pi_x, \Pi_y]_-$

> $\vec{\Pi} = \Pi[x] \cdot \_i + \Pi[y] \cdot \_j;$

$$\vec{\Pi} = \hat{i}\,\Pi_x + \hat{j}\,\Pi_y \quad \textbf{(6.2.1.9)}$$

> $\vec{p} = p[x] \cdot \_i + p[y] \cdot \_j$

$$\vec{p} = \hat{i}\,p_x + \hat{j}\,p_y \quad \textbf{(6.2.1.10)}$$

> **(6.2.1.3)**

$$\vec{\Pi} = \vec{p} - \frac{q\,\vec{A}}{c} \quad \textbf{(6.2.1.11)}$$

> $subs(\textbf{(6.2.1.6)}, \textbf{(6.2.1.9)}, \textbf{(6.2.1.10)}, \%)$

$$\hat{i}\,\Pi_x + \hat{j}\,\Pi_y = \hat{i}\,p_x + \hat{j}\,p_y - \frac{q\left(-\frac{1}{2}\,B\,\hat{i}\,y + \frac{1}{2}\,B\,\hat{j}\,x\right)}{c} \quad \textbf{(6.2.1.12)}$$

> $Component(\textbf{(6.2.1.12)}, 1)$

$$\Pi_x = p_x + \frac{q\,B\,y}{2\,c} \quad \textbf{(6.2.1.13)}$$

> $Component(\textbf{(6.2.1.12)}, 2)$

$$\Pi_y = p_y - \frac{q\,B\,x}{2\,c} \quad \textbf{(6.2.1.14)}$$

> $Commutator(\textbf{(6.2.1.13)}, \textbf{(6.2.1.14)})$

$$[\Pi_x, \Pi_y]_- = \frac{\mathrm{i}\,q\,B\,\hbar}{c} \quad \textbf{(6.2.1.15)}$$

> $Setup(\textbf{(6.2.1.15)})$

$$\left[algebrarules = \left\{[x, p_x]_- = \mathrm{i}\,\hbar,\ [x, p_y]_- = 0,\ [y, x]_- = 0,\ [y, p_x]_- = 0,\ [y, p_y]_- = \mathrm{i}\,\hbar,\ [\Pi_x, \Pi_y]_- = \frac{\mathrm{i}\,q\,B\,\hbar}{c},\ [p_y, p_x]_- = 0\right\}\right] \quad \textbf{(6.2.1.16)}$$

Time to bring in annihilation and creation operators

> $a = \frac{\sqrt{c}}{\sqrt{2 \cdot \hbar \cdot q \cdot B}} \left(\Pi_x + \mathrm{i} \cdot \Pi_y\right)$

$$a = \frac{\sqrt{c}\,\sqrt{2}\,\left(\Pi_x + \mathrm{i}\,\Pi_y\right)}{2\,\sqrt{\hbar\,q\,B}} \tag{6.2.1.17}$$

> **(6.2.1.17)***

$$a^{\dagger} = \frac{\sqrt{c}\,\sqrt{2}\,\left(\Pi_x - \mathrm{i}\,\Pi_y\right)}{2\,\sqrt{\hbar\,q\,B}} \tag{6.2.1.18}$$

Verify the normalization of this definition

> *Commutator*(**(6.2.1.17)**, **(6.2.1.18)**)

$$\left[a, a^{\dagger}\right]_{-} = 1 \tag{6.2.1.19}$$

> *Setup*(**(6.2.1.19)**)

$$\left[\mathit{algebrarules} = \left\{\left[a, a^{\dagger}\right]_{-} = 1, \left[x, p_x\right]_{-} = \mathrm{i}\,\hbar, \left[x, p_y\right]_{-} = 0, \left[y, x\right]_{-} = 0, \left[y, p_x\right]_{-} = 0, \left[y, p_y\right]_{-} = \mathrm{i}\,\hbar, \left[\Pi_x, \Pi_y\right]_{-} = \frac{\mathrm{i}\,q\,B\,\hbar}{c}, \left[p_y, p_x\right]_{-} = 0\right\}\right] \tag{6.2.1.20}$$

To express the Hamiltonian in terms of a, $a^{\dagger}$

> **(6.2.1.2)**

$$H = \frac{\vec{\Pi}^{2}}{2\,m} \tag{6.2.1.21}$$

> *subs*(**(6.2.1.9)**, %)

$$H = \frac{\left(\hat{i}\,\Pi_x + \hat{j}\,\Pi_y\right)^2}{2\,m} \tag{6.2.1.22}$$

> {**(6.2.1.17)**, **(6.2.1.18)**}

$$\left\{a = \frac{\sqrt{c}\,\sqrt{2}\,\left(\Pi_x + \mathrm{i}\,\Pi_y\right)}{2\,\sqrt{\hbar\,q\,B}}, a^{\dagger} = \frac{\sqrt{c}\,\sqrt{2}\,\left(\Pi_x - \mathrm{i}\,\Pi_y\right)}{2\,\sqrt{\hbar\,q\,B}}\right\} \tag{6.2.1.23}$$

> *solve*$\left(\%, \left\{\Pi_x, \Pi_y\right\}\right)$

$$\left\{\Pi_x = \frac{\sqrt{\hbar\,q\,B}\,\left(a^{\dagger} + a\right)\sqrt{2}}{2\,\sqrt{c}}, \Pi_y = \frac{\frac{\mathrm{i}}{2}\,\sqrt{\hbar\,q\,B}\,\left(a^{\dagger} - a\right)\sqrt{2}}{\sqrt{c}}\right\} \tag{6.2.1.24}$$

> *subs*(**(6.2.1.24)**, **(6.2.1.22)**)

$$H = \frac{\left(\frac{\hat{i}\,\sqrt{\hbar\,q\,B}\,\left(a^{\dagger} + a\right)\sqrt{2}}{2\,\sqrt{c}} + \frac{\mathrm{i}\,\hat{j}\,\sqrt{\hbar\,q\,B}\,\left(a^{\dagger} - a\right)\sqrt{2}}{2\,\sqrt{c}}\right)^2}{2\,m} \tag{6.2.1.25}$$

> *simplify*(*expand*(**(6.2.1.25)**))

(6.2.1.26)

$$H = \frac{\hbar q B \left(-1 + 2 a a^{\dagger}\right)}{2 m c} \quad \textbf{(6.2.1.26)}$$

> *Library:-SortProducts*(**(6.2.1.26)**, [*Dagger*(*a*), *a*], *usecommutator*)

$$H = \frac{\hbar q B \left(1 + 2 a^{\dagger} a\right)}{2 m c} \quad \textbf{(6.2.1.27)}$$

This is the Hamiltonian of an harmonic oscillator with frequency $\omega_c = \frac{q B}{m}$. The possible values for the energy are known: $E = \hbar\, \omega_c \left(n + \frac{1}{2}\right)$, where n is a positive integer.

>

Unitary Operators in Quantum Mechanics

*Eigenvalues of an unitary operator and exponential of Hermitian operators

- Show that the eigenvalues of an unitary operator are all on the unit circle, their modulus is 1.
- Show that an operator $e^{i \lambda H}$ is unitary provided H is Hermitian ($H = H^{\dagger}$) and λ is any real parameter.

> *restart*; *with*(*Physics*) : *interface*(*imaginaryunit* = *i*) :

> *Setup*(*unitaryoperators* = {*U*}, *quantumoperators* = {*V*}, *hermitianoperators* = {*H*}, *realobjects* = {λ})

[*hermitianoperators* = {*H*}, *quantumoperators* = {*H*, *U*, *V*}, *realobjects* = {λ}, *unitaryoperators* = {*U*}] **(6.3.1.1)**

If $\left| U_{\varepsilon} \right\rangle$ is a normalized eigenvector of U with eigenvalue ε

> $U \cdot Ket(U, \varepsilon) = U \cdot Ket(U, \varepsilon)$

$$U \left| U_{\varepsilon} \right\rangle = \varepsilon \left| U_{\varepsilon} \right\rangle \quad \textbf{(6.3.1.2)}$$

> *Dagger*(**(6.3.1.2)**)

$$\left\langle U_{\varepsilon} \right| U^{\dagger} = \bar{\varepsilon} \left\langle U_{\varepsilon} \right| \quad \textbf{(6.3.1.3)}$$

Multiplying sides by sides

> **(6.3.1.3)** . **(6.3.1.2)**

$$1 = |\varepsilon|^2 \quad \textbf{(6.3.1.4)}$$

Show that $e^{i \lambda H}$ is unitary

> *Library:-IsHermitianOperator*(*H*)

true **(6.3.1.5)**

> $V = \exp(i \cdot \lambda \cdot H)$

(6.3.1.6)

$$V = e^{i \lambda H} \quad (6.3.1.6)$$

> *Dagger*((6.3.1.6))

$$V^{\dagger} = e^{-i \lambda H} \quad (6.3.1.7)$$

> (6.3.1.6) . (6.3.1.7)

$$V V^{\dagger} = 1 \quad (6.3.1.8)$$

> (6.3.1.7) . (6.3.1.6)

$$V^{\dagger} V = 1 \quad (6.3.1.9)$$

Therefore, V is unitary

>

Properties of unitary operators

Consider two set of kets $| a_n \rangle$ and $| b_n \rangle$, each of them constituting a complete orthonormal basis of the same space.

One can always build an unitary operator U that maps one basis to the other, i.e.: $| b_n \rangle = U | a_n \rangle$

*Verify that $U = \sum_{k=0}^{\infty} | b_k \rangle \langle a_k |$ implies on $| b_n \rangle = U | a_n \rangle$

> *restart*; *with*(*Physics*) :

> *Setup*(*quantumoperators* = {*U*}, *bracketrules* = {%*Bracket*(*Bra*(*a*, *m*), *Ket*(*a*, *n*)) = *KroneckerDelta*[*m*, *n*], %*Bracket*(*Bra*(*b*, *m*), *Ket*(*b*, *n*)) = *KroneckerDelta*[*m*, *n*]}, *quantumdiscretebasis* = {*a*, *b*})

$$\left[bracketrules = \left\{ \langle a_m | a_n \rangle = \delta_{m, n}, \langle b_m | b_n \rangle = \delta_{m, n} \right\}, quantumdiscretebasis = \{a, b\}, quantumoperators = \{U\} \right] \quad (6.3.2.1.1)$$

> $U = \sum_{k=0}^{\infty} Ket(b, k)\, Bra(a, k)$

$$U = \sum_{k=0}^{\infty} | b_k \rangle \langle a_k | \quad (6.3.2.1.2)$$

Apply this operatorial equation to $| a_m \rangle$

> % . *Ket*(*a*, *m*)

$$U \bullet | a_m \rangle = | b_m \rangle \quad (6.3.2.1.3)$$

>

*Show that $U = \sum_{k=0}^{\infty} | b_k \rangle \langle a_k |$ is unitary

Recalling the expansion of the test operator U

> **(6.3.2.1.2)**

$$U = \sum_{k=0}^{\infty} |b_k\rangle\langle a_k| \quad \textbf{(6.3.2.2.1)}$$

> *Dagger*(**(6.3.2.1.2)**)

$$U^{\dagger} = \sum_{k=0}^{\infty} |a_k\rangle\langle b_k| \quad \textbf{(6.3.2.2.2)}$$

We have that

> **(6.3.2.2.2)** . **(6.3.2.1.2)**

$$U^{\dagger}\, U = \sum_{k1=0}^{\infty} |a_{k1}\rangle\langle a_{k1}| \quad \textbf{(6.3.2.2.3)}$$

> **(6.3.2.1.2)** . **(6.3.2.2.2)**

$$U\, U^{\dagger} = \sum_{k1=0}^{\infty} |b_{k1}\rangle\langle b_{k1}| \quad \textbf{(6.3.2.2.4)}$$

and since $|a_n\rangle$ and $|b_n\rangle$ form two complete basis, their Kets satisfy the closure relation, i. e. the right-hand side is equal to 1 and U is unitary.

>

*Show that the matrix elements of U in the $|a_n\rangle$ and $|b_n\rangle$ basis are equal

> **(6.3.2.1.2)**

$$U = \sum_{k=0}^{\infty} |b_k\rangle\langle a_k| \quad \textbf{(6.3.2.3.1)}$$

> *Bra*(a, n) . **(6.3.2.1.2)** . *Ket*(a, m)

$$\langle a_n | U | a_m \rangle = \langle a_n | b_m \rangle \quad \textbf{(6.3.2.3.2)}$$

Likewise

> *Bra*(b, n). **(6.3.2.1.2)** . *Ket*(b, m)

$$\langle b_n | U | b_m \rangle = \langle a_n | b_m \rangle \quad \textbf{(6.3.2.3.3)}$$

>

Show that A and $\mathcal{A} = U\, A\, U^{\dagger}$ have the same spectrum

> *Setup*(*redo*, *quantumoperators* = {A, $\mathcal{A}$}, *unitaryoperators* = {U})

$$[\mathit{quantumoperators} = \{\mathcal{A}, A, U\}, \mathit{unitaryoperators} = \{U\}] \quad \textbf{(6.3.2.4.1)}$$

If $\mathcal{A} = U\, A\, U^{\dagger}$, then the eigenkets of $\mathcal{A}$ are $|\mathcal{A}_{\alpha}\rangle = U\, |A_{\alpha}\rangle$

> $U . Ket(A, \alpha) = Ket(\mathcal{A}, \alpha)$

$$U \cdot |A_\alpha\rangle = |\mathcal{A}_\alpha\rangle \qquad \textbf{(6.3.2.4.2)}$$

> $U\, A\, Dagger(U) = \mathcal{A}$

$$U\, A\, U^\dagger = \mathcal{A} \qquad \textbf{(6.3.2.4.3)}$$

> **(6.3.2.4.3)** · **(6.3.2.4.2)**

$$U\, A\, U^\dagger \left(U \cdot |A_\alpha\rangle\right) = \mathcal{A} |\mathcal{A}_\alpha\rangle \qquad \textbf{(6.3.2.4.4)}$$

The left-hand side can be rewritten performing the product

> $lhs(\textbf{(6.3.2.4.4)}) = eval(lhs(\textbf{(6.3.2.4.4)}), `*` = `.`)$

$$U\, A\, U^\dagger \left(U \cdot |A_\alpha\rangle\right) = \alpha \left(U \cdot |A_\alpha\rangle\right) \qquad \textbf{(6.3.2.4.5)}$$

> $subs(\textbf{(6.3.2.4.5)}, \textbf{(6.3.2.4.4)})$

$$\alpha \left(U \cdot |A_\alpha\rangle\right) = \mathcal{A} |\mathcal{A}_\alpha\rangle \qquad \textbf{(6.3.2.4.6)}$$

> $subs(\textbf{(6.3.2.4.2)}, \textbf{(6.3.2.4.6)})$

$$\alpha |\mathcal{A}_\alpha\rangle = \mathcal{A} |\mathcal{A}_\alpha\rangle \qquad \textbf{(6.3.2.4.7)}$$

>

In conclusion, after an unitary transform, an eigenvector of the initial operator is an eigenvector of the new operator with the same eigenvalue.

Schrödinger equation and unitary transform

Consider a ket $|\psi_t\rangle$ that solves the time-dependant Schrödinger equation:

$$\mathrm{i}\, \hbar \frac{\partial}{\partial t} |\psi_t\rangle = H(t) |\psi_t\rangle$$

and consider

$$|\phi_t\rangle = U(t) |\psi_t\rangle,$$

where $U(t)$ is a unitary operator.

Does $|\phi_t\rangle$ evolves according a Schrödinger equation

$$\mathrm{i} \cdot \hbar \frac{\partial}{\partial t} |\phi_t\rangle = \mathcal{H}(t) |\phi_t\rangle$$

and if yes, which is the expression of $\mathcal{H}(t)$?

Solution

> *restart*; *with*(*Physics*) : *interface*(*imaginaryunit* = *i*) :

> *Setup*(*automaticsimplification* = *true*, *mathematicalnotation* = *true*, *quantumoperators* = {$\mathcal{H}$}, *hermitianoperators* = {*H*}, *unitaryoperators* = {*U*}, *realobjects* = {*t*, $\hbar$})

$$[automaticsimplification = true, hermitianoperators = \{H\}, mathematicalnotation = true, quantumoperators = \{\mathcal{H}, H, U\}, realobjects = \{\hbar, t\}, unitaryoperators = \{U\}] \quad \textbf{(6.3.3.1.1)}$$

> $CompactDisplay((U, H, \mathcal{H})(t))$

$U(t)$ *will now be displayed as* U

$H(t)$ *will now be displayed as* H

$\mathcal{H}(t)$ *will now be displayed as* $\mathcal{H}$ **(6.3.3.1.2)**

> $Ket(\phi, t) = U(t) \cdot Ket(\psi, t)$

$$|\phi_t\rangle = U|\psi_t\rangle \quad \textbf{(6.3.3.1.3)}$$

Compute now the evolution of $|\phi_t\rangle$

> $\mathrm{i} \cdot \hbar \cdot diff(\textbf{(6.3.3.1.3)}, t)$

$$\mathrm{i}\,\hbar\,|\phi_t\rangle_t = \mathrm{i}\,\hbar\,\left(U_t|\psi_t\rangle + U|\psi_t\rangle_t\right) \quad \textbf{(6.3.3.1.4)}$$

Simplify this equation taking into account Schrödinger's equation for ψ :

> $\mathrm{i} \cdot \hbar \frac{\partial}{\partial t} Ket(\psi, t) = H(t)\, Ket(\psi, t)$

$$\mathrm{i}\,\hbar\,|\psi_t\rangle_t = H|\psi_t\rangle \quad \textbf{(6.3.3.1.5)}$$

> $simplify\left(\textbf{(6.3.3.1.4)}, \{\textbf{(6.3.3.1.5)}\}, \left\{\frac{\partial}{\partial t} Ket(\psi, t)\right\}\right)$

$$\mathrm{i}\,\hbar\,|\phi_t\rangle_t = \mathrm{i}\,\hbar\,U_t|\psi_t\rangle + U\,H|\psi_t\rangle \quad \textbf{(6.3.3.1.6)}$$

Now, from

> **(6.3.3.1.3)**

$$|\phi_t\rangle = U|\psi_t\rangle \quad \textbf{(6.3.3.1.7)}$$

> $U(t)^* \cdot (rhs = lhs)(\textbf{(6.3.3.1.3)})$

$$U^\dagger\,U|\psi_t\rangle = U^\dagger|\phi_t\rangle \quad \textbf{(6.3.3.1.8)}$$

> $simplify(\textbf{(6.3.3.1.8)})$

$$|\psi_t\rangle = U^\dagger|\phi_t\rangle \quad \textbf{(6.3.3.1.9)}$$

Inserting this result in **(6.3.3.1.6)**

> $subs(\textbf{(6.3.3.1.9)}, \textbf{(6.3.3.1.6)})$

$$\mathrm{i}\,\hbar\,|\phi_t\rangle_t = \mathrm{i}\,\hbar\,U_t\left(U^\dagger|\phi_t\rangle\right) + U\,H\left(U^\dagger|\phi_t\rangle\right) \quad \textbf{(6.3.3.1.10)}$$

the $\mathcal{H}$amiltonian for $|\phi_t\rangle$ is given by the coefficient of $|\phi_t\rangle$ on the right-hand side

> $\mathcal{H}(t) = Coefficients(\mathrm{rhs}(\textbf{(6.3.3.1.10)}), Ket(\phi, t))$

$$\mathcal{H} = \mathrm{i}\,\hbar\,U_t\,U^\dagger + U\,H\,U^\dagger \quad \textbf{(6.3.3.1.11)}$$

So $|\phi_t\rangle$ satisfies a Schrodinger equation and as one can expect, $\mathcal{H}$ is Hermitian

> $Dagger((6.3.3.1.11)) - (6.3.3.1.11)$

$$\mathcal{H}^\dagger - \mathcal{H} = -\mathrm{i}\,\hbar\left(U\,{U^\dagger}_t + U_t\,U^\dagger\right) \quad (6.3.3.1.12)$$

Recalling that $U(t)$ satisfies

> $U(t)\cdot U(t)^* = U(t) \cdot U(t)^*$

$$U\,U^\dagger = 1 \quad (6.3.3.1.13)$$

> $diff((6.3.3.1.13), t)$

$$U\,{U^\dagger}_t + U_t\,U^\dagger = 0 \quad (6.3.3.1.14)$$

> $subs((6.3.3.1.14), (6.3.3.1.12))$

$$\mathcal{H}^\dagger - \mathcal{H} = 0 \quad (6.3.3.1.15)$$

In the time independent case, i.e. $U(t) = U$, $\mathcal{H}$ reduced to:

> $subs(U(t) = U, (6.3.3.1.11))$

$$\mathcal{H} = \mathrm{i}\,\hbar\,U_t\,U^\dagger + U\,H\,U^\dagger \quad (6.3.3.1.16)$$

> $\%$

$$\mathcal{H} = U\,H\,U^\dagger \quad (6.3.3.1.17)$$

>

Translation operators using Dirac notation

In this section, we focus on the operator $T_a = \mathrm{e}^{\frac{-\mathrm{i}\,a\,P}{\hbar}}$

Settings

> $restart; with(Physics) : interface(imaginaryunit = i) :$

> $Setup\left(realobjects = \{a, x, \hbar, m, x_1, x_2\}, unitaryoperators = \{T\}, hermitianoperators = \{\mathbb{I}, X, P\}, quantumcontinuousbasis = \{X, P\}\right)$

$$\left[hermitianoperators = \{\mathbb{I}, P, X\}, quantumcontinuousbasis = \{P, X\}, realobjects = \{\hbar, a, m, x, x_1, x_2\}, unitaryoperators = \{T\}\right] \quad (6.3.4.1.1)$$

> $Setup\left(bracketrules = \left\{Bracket(Bra(P, p), Ket(\psi)) = \tilde{\psi}(p), Bracket(Bra(X, x), Ket(\psi)) = \psi(x), Bracket(Bra(X, x), Ket(P, p)) = (2\cdot\pi\cdot\hbar)^{-\frac{1}{2}}\cdot\exp\left(\frac{\mathrm{i}}{\hbar}\cdot x\cdot p\right)\right\}\right)$

$$\left[bracketrules = \left\{\langle P_p \mid \psi\rangle = \tilde{\psi}(p), \langle X_x \mid \psi\rangle = \psi(x), \langle X_x \mid P_p\rangle\right.\right. \quad (6.3.4.1.2)$$

$$= \frac{\sqrt{2}\, e^{\frac{i x p}{\hbar}}}{2\sqrt{\pi \hbar}} \Big\} \Big] \Big]$$

> *Assume*(ℏ > 0)

$$\{\hbar::(0, \infty]\} \qquad \textbf{(6.3.4.1.3)}$$

Useful closure relations

> 1 = *Projector*(*Ket*(*X*, *x*))

$$1 = \int_{-\infty}^{\infty} |X_x\rangle \langle X_x| \,dx \qquad \textbf{(6.3.4.1.4)}$$

To have equivalent projectors with different integration variables, we use 𝕀 as the identity operator: $\mathbb{I}^{-1} = \mathbb{I}$, $\mathbb{I}_1 = \mathbb{I}_2$:

> 𝕀[1] = *Projector*(*Ket*(*P*, *p*)), 𝕀[2] = *Projector*(*Ket*(*P*, *q*))

$$\mathbb{I}_1 = \int_{-\infty}^{\infty} |P_p\rangle \langle P_p| \,dp,\ \mathbb{I}_2 = \int_{-\infty}^{\infty} |P_q\rangle \langle P_q| \,dq \qquad \textbf{(6.3.4.1.5)}$$

>

The Action (translation) of the operator $T_a = e^{-i \frac{a P}{\hbar}}$ on a ket

Considering a general ket $|\psi\rangle$, introduce a closure relation

> *Ket*(ψ) = $\mathbb{I}_1$ · *Ket*(ψ)

$$|\psi\rangle = \mathbb{I}_1 |\psi\rangle \qquad \textbf{(6.3.4.2.1)}$$

> *subs*(**(6.3.4.1.5)**, %)

$$|\psi\rangle = \left(\int_{-\infty}^{\infty} |P_p\rangle \langle P_p| \,dp\right) |\psi\rangle \qquad \textbf{(6.3.4.2.2)}$$

> *Bra*(*X*, *x*) . %

$$\psi(x) = \int_{-\infty}^{\infty} \frac{\sqrt{2}\sqrt{\frac{1}{\pi \hbar}}\, e^{\frac{i x p}{\hbar}}\, \tilde{\psi}(p)}{2} \,dp \qquad \textbf{(6.3.4.2.3)}$$

Which gives after a variable change $x = y - a$

> *PDEtools*:-*dchange*(*x* = *y* − *a*, %, {*y*}, *known* = ψ) : *subs*(*y* = *x*, %)

$$\psi(x - a) = \int_{-\infty}^{\infty} \frac{\sqrt{2}\sqrt{\frac{1}{\pi \hbar}}\, e^{\frac{i (x - a) p}{\hbar}}\, \tilde{\psi}(p)}{2} \,dp \qquad \textbf{(6.3.4.2.4)}$$

Let's now evaluate the action of $e^{-i \frac{a P}{\hbar}}$ on $|\psi\rangle$ in the $|X, x\rangle$ basis

> **(6.3.4.2.2)**

$$| \psi \rangle = \left(\int_{-\infty}^{\infty} | P_p \rangle \langle P_p | \, dp \right) | \psi \rangle \quad \textbf{(6.3.4.2.5)}$$

> *Bra*(*X*, *x*) . $e^{-\mathrm{i} \frac{a P}{\hbar}}$. %

$$\left\langle X_x \middle| e^{\frac{-\mathrm{i}\, a P}{\hbar}} \middle| \psi \right\rangle = \int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}}\, e^{\frac{-\mathrm{i}\, p (a - x)}{\hbar}} \tilde{\psi}(p)}{2} \, dp \quad \textbf{(6.3.4.2.6)}$$

Comparing the above with **(6.3.4.2.4)**

> % - **(6.3.4.2.4)**

$$\left\langle X_x \middle| e^{\frac{-\mathrm{i}\, a P}{\hbar}} \middle| \psi \right\rangle - \psi(x - a) = \int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}}\, e^{\frac{-\mathrm{i}\, p (a - x)}{\hbar}} \tilde{\psi}(p)}{2} \, dp - \left(\int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}}\, e^{\frac{\mathrm{i}\, (x - a) p}{\hbar}} \tilde{\psi}(p)}{2} \, dp \right) \quad \textbf{(6.3.4.2.7)}$$

> *simplify*(**(6.3.4.2.7)**)

$$\left\langle X_x \middle| e^{\frac{-\mathrm{i}\, a P}{\hbar}} \middle| \psi \right\rangle - \psi(x - a) = 0 \quad \textbf{(6.3.4.2.8)}$$

> *isolate*(%, ψ(*x* − *a*))

$$\psi(x - a) = \left\langle X_x \middle| e^{\frac{-\mathrm{i}\, a P}{\hbar}} \middle| \psi \right\rangle \quad \textbf{(6.3.4.2.9)}$$

>

Action of T_a on an operator $V(X)$

Let's consider an operator $V(X)$, that can be written as a formal power series:

$V(x) = \sum_{n=0}^{\infty} v_n \cdot x^n.$

Its matrix elements are:

> (*%Bracket* = *Bracket*)(*Bra*(*X*, x_1), *V*(*X*), *Ket*(*X*, x_2))

$$\left\langle X_{x_1} \middle| V(X) \middle| X_{x_2} \right\rangle = V(x_2)\, \delta(x_2 - x_1) \quad \textbf{(6.3.4.3.1)}$$

Using the closure relation

> **(6.3.4.1.4)**

$$1 = \int_{-\infty}^{\infty} | X_x \rangle \langle X_x | \, dx \quad \textbf{(6.3.4.3.2)}$$

$V(X)$ can also be represented in the $| X, x \rangle$ basis as

> $V(X) \cdot$ **(6.3.4.1.4)**

$$V(X) = \int_{-\infty}^{\infty} V(x) \left| X_x \right\rangle \left\langle X_x \right| \mathrm{d}x \qquad \textbf{(6.3.4.3.3)}$$

Let's now introduce two closure relations to evaluate $V(X)$ in the momentum basis $| P \rangle$

> $\left[Ket(X, x) = \mathbb{I}_1 \cdot Ket(X, x),\quad Bra(X, x) = Bra(X, x) \cdot \mathbb{I}_2\right]$

$$\left[\left| X_x \right\rangle = \mathbb{I}_1 \left| X_x \right\rangle, \left\langle X_x \right| = \left\langle X_x \right| \mathbb{I}_2\right] \qquad \textbf{(6.3.4.3.4)}$$

> $subs(\%, \textbf{(6.3.4.3.3)})$

$$V(X) = \int_{-\infty}^{\infty} V(x)\, \mathbb{I}_1 \left| X_x \right\rangle \left(\left\langle X_x \right| \mathbb{I}_2\right) \mathrm{d}x \qquad \textbf{(6.3.4.3.5)}$$

Recalling

> **(6.3.4.1.5)**

$$\mathbb{I}_1 = \int_{-\infty}^{\infty} \left| P_p \right\rangle \left\langle P_p \right| \mathrm{d}p, \mathbb{I}_2 = \int_{-\infty}^{\infty} \left| P_q \right\rangle \left\langle P_q \right| \mathrm{d}q \qquad \textbf{(6.3.4.3.6)}$$

> $subs(\%, \%\%)$

$$V(X) = \int_{-\infty}^{\infty} V(x) \left(\int_{-\infty}^{\infty} \left| P_p \right\rangle \left\langle P_p \right| \mathrm{d}p\right) \left| X_x \right\rangle \left\langle X_x \right| \left(\int_{-\infty}^{\infty} \left| P_q \right\rangle \left\langle P_q \right| \mathrm{d}q\right) \mathrm{d}x \qquad \textbf{(6.3.4.3.7)}$$

> $combine(\textbf{(6.3.4.3.7)})$

$$V(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(x) \left| P_p \right\rangle \left\langle P_p \right| \left| X_x \right\rangle \left\langle X_x \right| \left| P_q \right\rangle \left\langle P_q \right| \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}x \qquad \textbf{(6.3.4.3.8)}$$

> $eval(\%, `*` = `.`)$

$$V(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V(x)\, \mathrm{e}^{\frac{-\mathrm{i}\, x\, (p - q)}{\hbar}} \left| P_p \right\rangle \left\langle P_q \right|}{2\, \pi\, \hbar} \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}x \qquad \textbf{(6.3.4.3.9)}$$

Apply now the translation operator T_a

> $T[a] = \exp\left(-\frac{\mathrm{i}}{\hbar} \cdot a \cdot P\right)$

$$T_a = \mathrm{e}^{\frac{-\mathrm{i}\, a\, P}{\hbar}} \qquad \textbf{(6.3.4.3.10)}$$

> $\%\, .\, \textbf{(6.3.4.3.9)}\, .\, \%^*$

$$T_a\, V(X)\, T_a^{\dagger} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V(x) \left| P_p \right\rangle \left\langle P_q \right| \mathrm{e}^{\frac{-\mathrm{i}\, (p - q)\, (a + x)}{\hbar}}}{2\, \pi\, \hbar} \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}x \qquad \textbf{(6.3.4.3.11)}$$

Making a variable change $x = y - a$

> $PDEtools{:}\text{-}dchange(x = y - a, \%, \{y\}, known = V) : subs(y = x, \%)$

(6.3.4.3.12)

$$T_a\, V(X)\, T_a^{\dagger} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V(x-a)\, |P_p\rangle \langle P_q|\, e^{\frac{-i\, x\,(p-q)}{\hbar}}}{2\,\pi\,\hbar}\, dq\, dp\, dx \quad \textbf{(6.3.4.3.12)}$$

Evaluate the matrix element of this result and compute the integral

> $Bra(X, x_1) \cdot \% \cdot Ket(X, x_2)$

$$\langle X_{x_1} | T_a\, V(X)\, T_a^{\dagger} | X_{x_2} \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V(x-a)\, e^{\frac{i\,(-x+x_1)\,p + i\,q\,(x-x_2)}{\hbar}}}{4\,\pi^2\,\hbar^2}\, dq\, dp\, dx \quad \textbf{(6.3.4.3.13)}$$

> $value(\%)$

$$\langle X_{x_1} | T_a\, V(X)\, T_a^{\dagger} | X_{x_2} \rangle = V(x_1 - a)\, \delta(x_2 - x_1) \quad \textbf{(6.3.4.3.14)}$$

>

General Relativity

*Exact Solutions to Einstein's Equations $g_{\mu,\nu}\Lambda + G_{\mu,\nu} = 8\,\pi\,T_{\mu,\nu}$

Main reference: **-** Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. Herlt, E. **Exact Solutions of Einstein's Field Equations**, Cambridge Monographs on Mathematical Physics, second edition. Cambridge University Press, 2003.

*The authors reviewed more than **4,000 papers containing solutions to Einstein's equations** in the literature and organized the material into chapters according to the physical properties of these solutions.*

These solutions are now digitized within Maple 2016, so that ***it is now possible to actually compute with them***.

- The solutions are turned active by a simple call to the g_ spacetime metric.
- Everything else gets automatically derived on the fly (Christoffel symbols , Ricci and Riemann tensors orthonormal and null tetrads, etc.)
- Almost all of the mathematical operations one can perform on these solutions are implemented as commands in the Physics and DifferentialGeometry packages.
- All the mathematics within the Maple library are readily available to work with these solutions.

Examples

Load Physics, set the metric to be *Schwarzschild's* solution (and everything else automatically) in one go

> $restart; with(Physics):$

> $g\_[sc]$

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu,\nu} = \begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \quad \textbf{(7.1.1.1)}$$

And that is all we do.

The tensor components of the general relativity tensors related to this solution get derived automatically.

> $Riemann[definition]$

$$R_{\mu,\nu,\alpha,\beta} = g_{\mu,\lambda}\left(\partial_\alpha\left(\Gamma^{\lambda}{}_{\nu,\beta}\right) - \left(\partial_\beta\left(\Gamma^{\lambda}{}_{\nu,\alpha}\right)\right) + \Gamma^{\lambda}{}_{\kappa,\alpha}\Gamma^{\kappa}{}_{\nu,\beta} - \Gamma^{\lambda}{}_{\kappa,\beta}\Gamma^{\kappa}{}_{\nu,\alpha}\right) \quad \textbf{(7.1.1.2)}$$

The invariants using the formulas by Carminati and McLenaghan

> $Riemann[invariants]$

$$r_0 = 0, r_1 = 0, r_2 = 0, r_3 = 0, w_1 = \frac{6m^2}{r^6}, w_2 = \frac{6m^3}{r^9}, m_1 = 0, m_2 = 0, m_3 = 0, m_4 = 0, m_5 = 0 \quad \textbf{(7.1.1.3)}$$

The related Weyl scalars in the context of the Newman-Penrose formalism

> $Weyl[scalarsdefinition]$

$$\psi_0 = -C^{\mu,\nu,\alpha,\beta} l_\mu m_\nu l_\alpha m_\beta, \psi_1 = -C^{\mu,\nu,\alpha,\beta} l_\mu n_\nu l_\alpha m_\beta, \psi_2 = -C^{\mu,\nu,\alpha,\beta} l_\mu m_\nu \overline{m}_\alpha n_\beta, \psi_3 = -C^{\mu,\nu,\alpha,\beta} l_\mu n_\nu \overline{m}_\alpha n_\beta, \psi_4 = -C^{\mu,\nu,\alpha,\beta} n_\mu \overline{m}_\nu n_\alpha \overline{m}_\beta \quad \textbf{(7.1.1.4)}$$

> $Weyl[scalars]$

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = -\frac{m}{r^3}, \psi_3 = 0, \psi_4 = 0 \quad \textbf{(7.1.1.5)}$$

The four Killing vectors

> $Define(K, quiet):$

> *KillingVectors*(*K*)

$$\left[K^{\mu} = [0, 0, 0, 1], K^{\mu} = \left[0, \sin(\phi), \frac{\cos(\phi)}{\tan(\theta)}, 0\right], K^{\mu} = \left[0, \cos(\phi), -\frac{\sin(\phi)}{\tan(\theta)}, 0\right], K^{\mu} = [0, 0, 1, 0]\right] \quad (7.1.1.6)$$

These are the 2x2 matrix components of the Christoffel symbols of the second kind

> *Christoffel*[~1, alpha, beta, *matrix*]

$$\Gamma^{1}{}_{\alpha, \beta} = \begin{bmatrix} \frac{m}{r(-r+2m)} & 0 & 0 & 0 \\ 0 & -r+2m & 0 & 0 \\ 0 & 0 & (-r+2m)\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{-2m^2+mr}{r^3} \end{bmatrix} \quad (7.1.1.7)$$

This is the tetrad related to Schwarzschild's solution

> *with*(*Tetrads*) :

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (*see ?Physics,tetrads*), $e_{a, \mu}, \eta_{a, b}, \gamma_{a, b, c}, \lambda_{a, b, c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (*see ?Physics,tetrads*), $l_{\mu}, n_{\mu}, m_{\mu}, \overline{m}_{\mu}$ **(7.1.1.8)**

> *e*_[ ]

$$e_{a, \mu} = \begin{bmatrix} \frac{-I\sqrt{r}}{\sqrt{-r+2m}} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin(\theta) & 0 \\ 0 & 0 & 0 & \frac{-I\sqrt{-r+2m}}{\sqrt{r}} \end{bmatrix} \quad (7.1.1.9)$$

One can check these things directly; for instance this is the definition of the tetrad, where the right-hand side is the tetrad metric

> *e*_[*definition*]

$$e_{a, \mu} e_{b}{}^{\mu} = \eta_{a, b} \quad (7.1.1.10)$$

This shows that, for the components given by **(7.1.1.9)**, the definition holds

> *TensorArray*(**(7.1.1.10)**)

(7.1.1.11)

$$\left[\begin{array}{cccc} -1=-1 & 0=0 & 0=0 & 0=0 \\ 0=0 & -1=-1 & 0=0 & 0=0 \\ 0=0 & 0=0 & -1=-1 & 0=0 \\ 0=0 & 0=0 & 0=0 & 1=1 \end{array}\right] \quad \textbf{(7.1.1.11)}$$

These are Einstein's equations

> *Einstein*[mu, nu] = 8 Pi *T*[mu, nu]

$$G_{\mu,\nu} = 8\,\pi\,T_{\mu,\nu} \quad \textbf{(7.1.1.12)}$$

where the right-hand side is the energy momentum tensor.

Departing from this expression, verify that all the components of $T_{\mu,\nu}$ are equal to 0 , i.e. that the Schwarzschild metric is a Schwarzschild vacuum solution.

> *Einstein*[*definition*]

$$G_{\mu,\nu} = R_{\mu,\nu} - \frac{g_{\mu,\nu}\,R_{\alpha}^{\ \alpha}}{2} \quad \textbf{(7.1.1.13)}$$

Substituting in **(7.1.1.12)**

> *isolate*(*subs*(*Einstein*[*definition*], **(7.1.1.12)**), *T*[mu, nu])

$$T_{\mu,\nu} = -\frac{-R_{\mu,\nu} + \frac{g_{\mu,\nu}\,R_{\alpha}^{\ \alpha}}{2}}{8\,\pi} \quad \textbf{(7.1.1.14)}$$

> *convert*(**(7.1.1.14)**, *g_*)

$$T_{\mu,\nu} = \frac{1}{8\,\pi}\left(\frac{\left(\partial_{\alpha}\left(g^{\alpha,\tau}\right)\right)\left(\partial_{\nu}\left(g_{\mu,\tau}\right)+\partial_{\mu}\left(g_{\nu,\tau}\right)-\left(\partial_{\tau}\left(g_{\mu,\nu}\right)\right)\right)}{2} + \frac{g^{\alpha,\tau}\left(\partial_{\alpha}\left(\partial_{\nu}\left(g_{\mu,\tau}\right)\right)+\partial_{\alpha}\left(\partial_{\mu}\left(g_{\nu,\tau}\right)\right)-\left(\partial_{\alpha}\left(\partial_{\tau}\left(g_{\mu,\nu}\right)\right)\right)\right)}{2} - \frac{\left(\partial_{\nu}\left(g^{\alpha,\lambda}\right)\right)\left(\partial_{\mu}\left(g_{\alpha,\lambda}\right)\right)}{2} - \frac{g^{\alpha,\lambda}\left(\partial_{\nu}\left(\partial_{\mu}\left(g_{\alpha,\lambda}\right)\right)\right)}{2} + \frac{g^{\beta,\chi}\left(\partial_{\nu}\left(g_{\chi,\mu}\right)+\partial_{\mu}\left(g_{\chi,\nu}\right)-\left(\partial_{\chi}\left(g_{\mu,\nu}\right)\right)\right)g^{\alpha,\kappa}\left(\partial_{\beta}\left(g_{\alpha,\kappa}\right)\right)}{4} - \frac{1}{4}\left(g^{\beta,\upsilon}\left(\partial_{\mu}\left(g_{\alpha,\upsilon}\right)+\partial_{\alpha}\left(g_{\mu,\upsilon}\right)-\left(\partial_{\upsilon}\left(g_{\alpha,\mu}\right)\right)\right)g^{\alpha,\sigma}\left(\partial_{\nu}\left(g_{\beta,\sigma}\right)+\partial_{\beta}\left(g_{\nu,\sigma}\right)-\left(\partial_{\sigma}\left(g_{\beta,\nu}\right)\right)\right)\right)\right) \quad \textbf{(7.1.1.15)}$$

Define this expression as a tensor, and compute its components

> *Define*((**7.1.1.15**))

Defined objects with tensor properties

$$\left\{\mathcal{D}_{\mu}, \gamma_{\mu}, K^{\mu}, \sigma_{\mu}, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_{\mu,\nu}, C_{\mu,\nu,\alpha,\beta}, X_{\mu}, \partial_{\mu}, e_{a,\mu}, \eta_{a,b}, g_{\mu,\nu}, \gamma_{a,b,c}, l_{\mu}, \lambda_{a,b,c}, m_{\mu}, \overline{m}_{\mu}, n_{\mu}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \quad \textbf{(7.1.1.16)}$$

> *T*[ ]

$$T_{\mu,\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(7.1.1.17)}$$

• One can query the database, directly from the spacetime metrics command (g_).

For example, these are the solutions (metrics) to Einstein's equations that appear in the book and related to Levi-Civita, the Italian mathematician

> *g_*[*civi*]

[12, 16, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)", "Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell", "SecondaryDescription" = ["Homogeneous"]]

[12, 18, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)", "Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell", "SecondaryDescription" = ["Homogeneous"]]

[12, 19, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)", "Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell", "SecondaryDescription" = ["Homogeneous"], "Comments" = ["_lambda=_zeta"]]

[22, 7, 1] = ["Authors" = ["Levi-Civita (1917), Frehland (1971)"], "PrimaryDescription" = "Vacuum", "SecondaryDescription" = ["Cylindrically-Symmetric"], "Comments" = ["Locally static, Weyl class_m=0,1 - flat, _m=1/2, 2, -1 - PetrovType D"]] **(7.1.1.18)**

These solutions can be set in one go from the metrics command, just by indicating the number with which it appears in the "Exact Solutions to Einstein's Equations" book.

The ability to query rapidly and set things in one go change the game.

• One can also query visually and by properties.

For example, search for solutions in the database that are of *Pure Radiation* type, with *Petrov Type* "*D*", *Plebanski-Petrov Type* "*O*" and that have *Isometry Dimension* equal to 1:

> *DifferentialGeometry:-Library:-MetricSearch*( )

Set one of these solutions and everything related in one go

> g_[[28, 74, 1]]

Systems of spacetime Coordinates are: $\{X = (u, \eta, r, y)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (u, \eta, r, y)\}$

The Frolov and Khlebnikov (1975) metric in coordinates $[u, \eta, r, y]$

Parameters: $[\kappa 0, m(u), b, d]$

Comments: With _m(u) = constant, the metric is Ricci flat and becomes 28.24 in Stephani.

Resetting the signature of spacetime from "- - - +" to `- + + +` in order to match the signature in the database of metrics:

$$g_{\mu,\nu} = \left[\left[\frac{2\,m(u)^3 - 6\,m(u)^2\,\eta\,r - r^2\,(-6\,\eta^2 + b)\,m(u) + r^3\,(-2\,\eta^3 + b\,\eta + d)}{r\,m(u)^2}, -\frac{r^2}{m(u)}, -1, 0\right], \left[-\frac{r^2}{m(u)}, \frac{r^2}{-2\,\eta^3 + b\,\eta + d}, 0, 0\right], [-1, 0, 0, 0], \left[0, 0, 0, r^2\,(-2\,\eta^3 + b\,\eta + d)\right]\right] \quad \textbf{(7.1.1.19)}$$

The amount of solutions/cases found in the book and digitized in Maple 2016

> *nops*(*DifferentialGeometry:-Library:-Retrieve*("Stephani", 1))

$$971 \quad \textbf{(7.1.1.20)}$$

>

*"Physical Review D" 87, 044053 (2013)

Given the spacetime metric,

$$g_{\mu,\nu} = \begin{bmatrix} -e^{\lambda(r)} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & e^{\nu(r)} \end{bmatrix}$$

a) Compute the Ricci and Weyl scalars

b) Compute the trace of

$$Z_{\alpha}^{\ \beta} = \Phi R_{\alpha}^{\ \beta} + \mathcal{D}_{\alpha} \mathcal{D}^{\beta} \Phi + T_{\alpha}^{\ \beta}$$

where $\Phi \equiv \Phi(r)$ is some function of the radial coordinate, $R_{\alpha}^{\ \beta}$ is the Ricci tensor, $\mathcal{D}_{\alpha}$ is the covariant derivative operator and $T_{\alpha}^{\ \beta}$ is the stress-energy tensor

$$T_{\alpha,\beta} = \begin{bmatrix} 8\, e^{\lambda(r)}\, \pi & 0 & 0 & 0 \\ 0 & 8\, r^2\, \pi & 0 & 0 \\ 0 & 0 & 8\, r^2 \sin(\theta)^2\, \pi & 0 \\ 0 & 0 & 0 & 8\, e^{\nu(r)}\, \pi\, \varepsilon \end{bmatrix}$$

c) Compute the components of $W_{\alpha}^{\ \beta} \equiv$ the traceless part of $Z_{\alpha}^{\ \beta}$ of item b**)**

d) Compute an exact solution to the nonlinear system of differential equations conformed by the components of $W_{\alpha}^{\ \beta}$ obtained in c**)**

Background: paper from February/2013, "Withholding Potentials, Absence of Ghosts and Relationship between Minimal Dilatonic Gravity and f(R) Theories", by P. Fiziev.

a) The Ricci and Weyl scalars

```
> restart; with(Physics) :
```

Set the coordinates

```
> Setup(coordinates = spherical, automaticsimplification = true)
```

** Partial match of 'coordinates' against keyword 'coordinatesystems'*

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

$[automaticsimplification = true, coordinatesystems = \{X\}]$ **(7.2.1.1)**

The square of the line element and the metric

> $ds^2 := \exp(\mathrm{nu}(r))dt^2 - \exp(\mathrm{lambda}(r))dr^2 - r^2\, dtheta^2 - r^2 \sin(\mathrm{theta})^2 dphi^2$

$$ds^2 := -e^{\lambda(r)}\, dr^2 + e^{\nu(r)}\, dt^2 - r^2 \left(d\phi^2 \sin(\theta)^2 + d\theta^2\right) \quad \textbf{(7.2.1.2)}$$

> $CompactDisplay(ds^2)$

$\lambda(r)$ *will now be displayed as* λ

$\nu(r)$ *will now be displayed as* ν **(7.2.1.3)**

> $Setup(metric = ds^2) : g\_[\ ]$

$$g_{\mu,\nu} = \begin{bmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & e^{\nu} \end{bmatrix} \quad \textbf{(7.2.1.4)}$$

> *with(Tetrads)*

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (see ?Physics,tetrads), $l_{\mu}, n_{\mu}, m_{\mu}, \overline{m}_{\mu}$

[IsTetrad, NullTetrad, OrthonormalTetrad, PetrovType, SimplifyTetrad, TransformTetrad, e_, eta_, gamma_, l_, lambda_, m_, mb_, n_] **(7.2.1.5)**

> *PetrovType( )*

"D" **(7.2.1.6)**

> *e_[nullvectors]*

$$l_{\mu} = \left[\frac{\sqrt{2}\, e^{\frac{\lambda}{2}}}{2}\ \ 0\ \ 0\ \ \frac{\sqrt{2}\, e^{\frac{\nu}{2}}}{2}\right], n_{\mu} = \left[-\frac{\sqrt{2}\, e^{\frac{\lambda}{2}}}{2}\ \ 0\ \ 0\ \ \frac{\sqrt{2}\, e^{\frac{\nu}{2}}}{2}\right], m_{\mu} = \left[0\ \ \frac{\sqrt{2}\, r}{2}\ \ \frac{I}{2}\sqrt{2}\, r\sin(\theta)\ \ 0\right], \overline{m}_{\mu} = \left[0\ \ \frac{\sqrt{2}\, r}{2}\ \ -\frac{I}{2}\sqrt{2}\, r\sin(\theta)\ \ 0\right] \quad \textbf{(7.2.1.7)}$$

> *Ricci[scalarsdefinition]*

$$\Phi_{00} = -R^{\mu,\nu} l_{\mu} l_{\nu}, \Phi_{01} = -R^{\mu,\nu} l_{\mu} m_{\nu}, \Phi_{02} = -R^{\mu,\nu} m_{\mu} m_{\nu}, \Phi_{11} = -R^{\mu,\nu}\left(l_{\mu} n_{\nu} + m_{\mu}\overline{m}_{\nu}\right), \Phi_{12} = -R^{\mu,\nu} n_{\mu} m_{\nu}, \Phi_{22} = -R^{\mu,\nu} n_{\mu} n_{\nu}, \Lambda = \frac{R_{\mu}^{\ \mu}}{24} \quad \textbf{(7.2.1.8)}$$

> *Ricci[scalars]*

$$\Phi_{00} = -\frac{e^{-\lambda}\left(\lambda_r + \nu_r\right)}{2\,r}, \Phi_{01} = 0, \Phi_{02} = 0, \Phi_{11} = \frac{-4 + \left(-\nu_r^{\,2}\,r^2 + \nu_r\,\lambda_r\,r^2 - 2\,\nu_{r,r}\,r^2 + 4\right)e^{-\lambda}}{4\,r^2}, \Phi_{12} = 0, \Phi_{22} = -\frac{e^{-\lambda}\left(\lambda_r + \nu_r\right)}{2\,r}, \Lambda = \frac{e^{-\lambda}\left(2\,\nu_{r,r}\,r^2 + \nu_r^{\,2}\,r^2 + \left(-r^2\,\lambda_r + 4\,r\right)\nu_r - 4\,\lambda_r\,r - 4\,e^{\lambda} + 4\right)}{48\,r^2} \tag{7.2.1.9}$$

> *Weyl*[*scalarsdefinition*]

$$\psi_0 = -C^{\mu,\nu,\alpha,\beta}\,l_\mu\,m_\nu\,l_\alpha\,m_\beta, \psi_1 = -C^{\mu,\nu,\alpha,\beta}\,l_\mu\,n_\nu\,l_\alpha\,m_\beta, \psi_2 = -C^{\mu,\nu,\alpha,\beta}\,l_\mu\,m_\nu\,\overline{m}_\alpha\,n_\beta, \psi_3 = -C^{\mu,\nu,\alpha,\beta}\,l_\mu\,n_\nu\,\overline{m}_\alpha\,n_\beta, \psi_4 = -C^{\mu,\nu,\alpha,\beta}\,n_\mu\,\overline{m}_\nu\,n_\alpha\,\overline{m}_\beta \tag{7.2.1.10}$$

> *Weyl*[*scalars*]

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = \frac{e^{-\lambda}\left(2\,\nu_{r,r}\,r^2 + \nu_r^{\,2}\,r^2 + \left(-r^2\,\lambda_r - 2\,r\right)\nu_r + 2\,\lambda_r\,r - 4\,e^{\lambda} + 4\right)}{24\,r^2}, \psi_3 = 0, \psi_4 = 0 \tag{7.2.1.11}$$

> *Setup*(*signature*)

$$[\mathit{signature} = - - - +] \tag{7.2.1.12}$$

>

b) The trace of $Z_\alpha^{\ \beta} = \Phi\,R_\alpha^{\ \beta} + \mathcal{D}_\alpha\,\mathcal{D}^\beta\,\Phi + T_\alpha^{\ \beta}$

The indicated stress-energy tensor

> T[alpha, beta] = 8·Pi·*Matrix*(4, ⟨exp(lambda(*r*)), r^2, r^2sin(theta)^2, epsilon exp(nu(*r*))⟩, *shape* = *diagonal*)

$$T_{\alpha,\beta} = \begin{bmatrix} 8\,\pi\,e^{\lambda} & 0 & 0 & 0 \\ 0 & 8\,\pi\,r^2 & 0 & 0 \\ 0 & 0 & 8\,\pi\,r^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & 8\,\pi\,\epsilon\,e^{\nu} \end{bmatrix} \tag{7.2.2.1}$$

> *Define*(**(7.2.2.1)**)

Defined objects with tensor properties **(7.2.2.2)**

$$\left\{\mathcal{D}_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_{\alpha,\beta}, C_{\mu,\nu,\alpha,\beta}, X_{\mu}, \partial_{\mu}, e_{a,\mu}, \eta_{a,b}, g_{\mu,\nu}, \gamma_{a,b,c}, l_{\mu}, \lambda_{a,b,c}, m_{\mu}, \overline{m}_{\mu}, n_{\mu}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \quad (7.2.2.2)$$

Solve item **b)** in one go, that is the trace of Z, defining the tensorial equation $Z_{\alpha}^{\ \beta} = \Phi R_{\alpha}^{\ \beta} + \mathcal{D}_{\alpha}\mathcal{D}^{\beta}\Phi + T_{\alpha}^{\ \beta}$

```
> CompactDisplay(Phi(r))
```

$$\Phi(r) \text{ will now be displayed as } \Phi \quad (7.2.2.3)$$

```
> Z[mu, nu] = Phi(r) Ricci[mu, nu] + 'D_[mu](D_[nu](Phi(r)))' + T[mu, nu]
```

$$Z_{\mu,\nu} = \Phi R_{\mu,\nu} + \mathcal{D}_{\mu}\left(\mathcal{D}_{\nu}(\Phi)\right) + T_{\mu,\nu} \quad (7.2.2.4)$$

```
> Define((7.2.2.4))
```

Defined objects with tensor properties

$$\left\{\mathcal{D}_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_{\alpha,\beta}, C_{\mu,\nu,\alpha,\beta}, X_{\mu}, Z_{\mu,\nu}, \partial_{\mu}, e_{a,\mu}, \eta_{a,b}, g_{\mu,\nu}, \gamma_{a,b,c}, l_{\mu}, \lambda_{a,b,c}, m_{\mu}, \overline{m}_{\mu}, n_{\mu}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \quad (7.2.2.5)$$

The answer to **a)**, that is the trace of $Z_{\mu,\nu}$

```
> Z[mu, mu]
```

$$Z_{\mu}^{\ \mu} \quad (7.2.2.6)$$

```
> SumOverRepeatedIndices((7.2.2.6))
```

$$\frac{1}{4r^2}\Big(-e^{-\nu} r\left(-2\nu_{r,r}\Phi r + \nu_{r}\left(r\lambda_{r}\Phi - \Phi\nu_{r} r + 2r\Phi_{r} - 4\Phi\right)\right)e^{-\lambda+\nu} + \left(2r^2\nu_{r,r}\Phi - 4r^2\Phi_{r,r} + r^2\Phi\nu_{r}^{\,2} - r\Phi\left(\lambda_{r} r - 4\right)\nu_{r} + \left(2r^2\Phi_{r} - 8r\Phi\right)\lambda_{r} - 32r^2 e^{\lambda}\pi - 8r\Phi_{r} + 8\Phi\right)e^{-\lambda} + 32e^{-\nu}\pi e^{\nu}\epsilon r^2 - 64\pi r^2 - 8\Phi\Big) \quad (7.2.2.7)$$

```
>
```

b) The components of $W_{\alpha}^{\ \beta} \equiv$ the traceless part of $Z_{\alpha}^{\ \beta}$

Define a tensor $W_{\mu,\nu}$ with the traceless part of $Z_{\mu,\nu}$

```
> W[mu, nu] = Z[mu, nu] - Z[alpha, alpha]/4 g_[mu, nu]
```

$$W_{\mu,\nu} = Z_{\mu,\nu} - \frac{Z_{\alpha}^{\ \alpha} g_{\mu,\nu}}{4} \quad (7.2.3.1)$$

```
> Define((7.2.3.1))
```

Defined objects with tensor properties

$$\left\{\mathcal{D}_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_{\alpha,\beta}, W_{\mu,\nu}, C_{\mu,\nu,\alpha,\beta}, X_{\mu}, Z_{\mu,\nu}, \partial_{\mu}, e_{a,\mu}, \eta_{a,b}, g_{\mu,\nu}, \right. \quad (7.2.3.2)$$

$\gamma_{a, b, c}, l_{\mu}, \lambda_{a, b, c}, m_{\mu}, \overline{m}_{\mu}, n_{\mu}, \Gamma_{\mu, \nu, \alpha}, G_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}\}$

Verify that W is traceless

> $W[\mathrm{mu}, \mathrm{mu}]$

$$W_{\mu}^{\ \mu} \tag{7.2.3.3}$$

> $SumOverRepeatedIndices((\mathbf{7.2.3.3}), simplifier = simplify)$

$$0 \tag{7.2.3.4}$$

The nonzero components for the traceless $W_{\mu}^{\ \nu}$ are then

> $W[\mathrm{mu}, \sim\mathrm{nu}, nonzero]$

$$W_{\mu}^{\ \nu} = \Big\{ (1,1) = \frac{1}{8 r^2} \Big(\Big(-6 r^2 \Phi_{r,r} + 2 r^2 \nu_{r,r} \Phi + r^2 \Phi \nu_r^{\,2} - r \big(r \lambda_r \Phi - r \Phi_r + 4 \Phi \big) \nu_r + \big(3 r^2 \Phi_r - 4 r \Phi \big) \lambda_r + 4 r \Phi_r - 4 \Phi \Big) e^{-\lambda} + 4 \Phi + (-16 \epsilon - 16) \pi r^2 \Big), (2,2) = \frac{1}{16 r^2} \Big(-16 - \frac{1}{16} e^{-\lambda} \nu_r \lambda_r \Phi r^2 - 16 - \frac{1}{8} e^{-\lambda} \nu_r \Phi_r r^2 - 16 \frac{1}{8} e^{-\lambda} \nu_{r,r} \Phi r^2 - 16 \frac{1}{4} e^{-\lambda} \nu_r \Phi r - 16\, 2 \pi \epsilon r^2 - 16 - 2 \pi r^2 + \Big(4 r^2 \Phi_{r,r} - 2 r^2 \nu_{r,r} \Phi - 2 r^2 \Phi \nu_r^{\,2} + \big(r^2 \Phi \lambda_r + 4 r \Phi \big) \nu_r - 2 \lambda_r \Phi_r r^2 - 8 r \Phi_r + 8 \Phi \Big) e^{-\lambda} - 64 \pi r^2 - 8 \Phi \Big), (3,3) = \frac{1}{16 r^2} \Big(- - e^{-\lambda} \nu_r \lambda_r \Phi r^2 - -2 e^{-\lambda} \nu_r \Phi_r r^2 - 2 e^{-\lambda} \nu_{r,r} \Phi r^2 - 4 e^{-\lambda} \nu_r \Phi r - 32 \pi \epsilon r^2 - -32 \pi r^2 + \Big(4 r^2 \Phi_{r,r} - 2 r^2 \nu_{r,r} \Phi - 2 r^2 \Phi \nu_r^{\,2} + \big(r^2 \Phi \lambda_r + 4 r \Phi \big) \nu_r - 2 \lambda_r \Phi_r r^2 - 8 r \Phi_r + 8 \Phi \Big) e^{-\lambda} - 64 \pi r^2 - 8 \Phi \Big), (4,4) = \frac{1}{8 r^2} \Big(\Big(2 r^2 \Phi_{r,r} + 2 r^2 \nu_{r,r} \Phi + r^2 \Phi \nu_r^{\,2} - r \big(r \lambda_r \Phi + 3 r \Phi_r - 4 \Phi \big) \nu_r + \big(-r^2 \Phi_r + 4 r \Phi \big) \lambda_r + 4 r \Phi_r - 4 \Phi \Big) e^{-\lambda} + 4 \Phi + (48 \epsilon + 48) \pi r^2 \Big) \Big\} \tag{7.2.3.5}$$

>

c) An exact solution for the nonlinear system of differential equations conformed by the components of $W_{\alpha}^{\ \beta}$

Create an ODE system with the nonzero components of $W_{\mu}^{\ \nu}$

> $ode_{system} := map(u \to rhs(u) = 0, rhs(\mathbf{(7.2.3.5)}))$

$$ode_{system} := \left\{ \frac{1}{8 r^2} \left(\left(2 r^2 \Phi_{r,r} - 2 r^2 \nu_{r,r} \Phi + \left(r^2 \nu_r - r^2 \lambda_r - 4 r \right) \Phi_r - \Phi \left(\nu_r^2 r^2 - \nu_r \lambda_r r^2 - 4 \right) \right) e^{-\lambda} - 4 \Phi + (-16 \epsilon - 16) \pi r^2 \right) = 0, \frac{1}{8 r^2} \left(\left(-6 r^2 \Phi_{r,r} + 2 r^2 \nu_{r,r} \Phi + r^2 \Phi \nu_r^2 - r \left(r \lambda_r \Phi - r \Phi_r + 4 \Phi \right) \nu_r + \left(3 r^2 \Phi_r - 4 r \Phi \right) \lambda_r + 4 r \Phi_r - 4 \Phi \right) e^{-\lambda} + 4 \Phi + (-16 \epsilon - 16) \pi r^2 \right) = 0, \frac{1}{8 r^2} \left(\left(2 r^2 \Phi_{r,r} + 2 r^2 \nu_{r,r} \Phi + r^2 \Phi \nu_r^2 - r \left(r \lambda_r \Phi + 3 r \Phi_r - 4 \Phi \right) \nu_r + \left(-r^2 \Phi_r + 4 r \Phi \right) \lambda_r + 4 r \Phi_r - 4 \Phi \right) e^{-\lambda} + 4 \Phi + (48 \epsilon + 48) \pi r^2 \right) = 0 \right\} \quad \mathbf{(7.2.4.1)}$$

Run a differential elimination process towards identifying singular cases, frequently simpler to solve: there are three cases

> $cases := [PDEtools\text{:-}casesplit(ode_{system}, caseplot)]:$

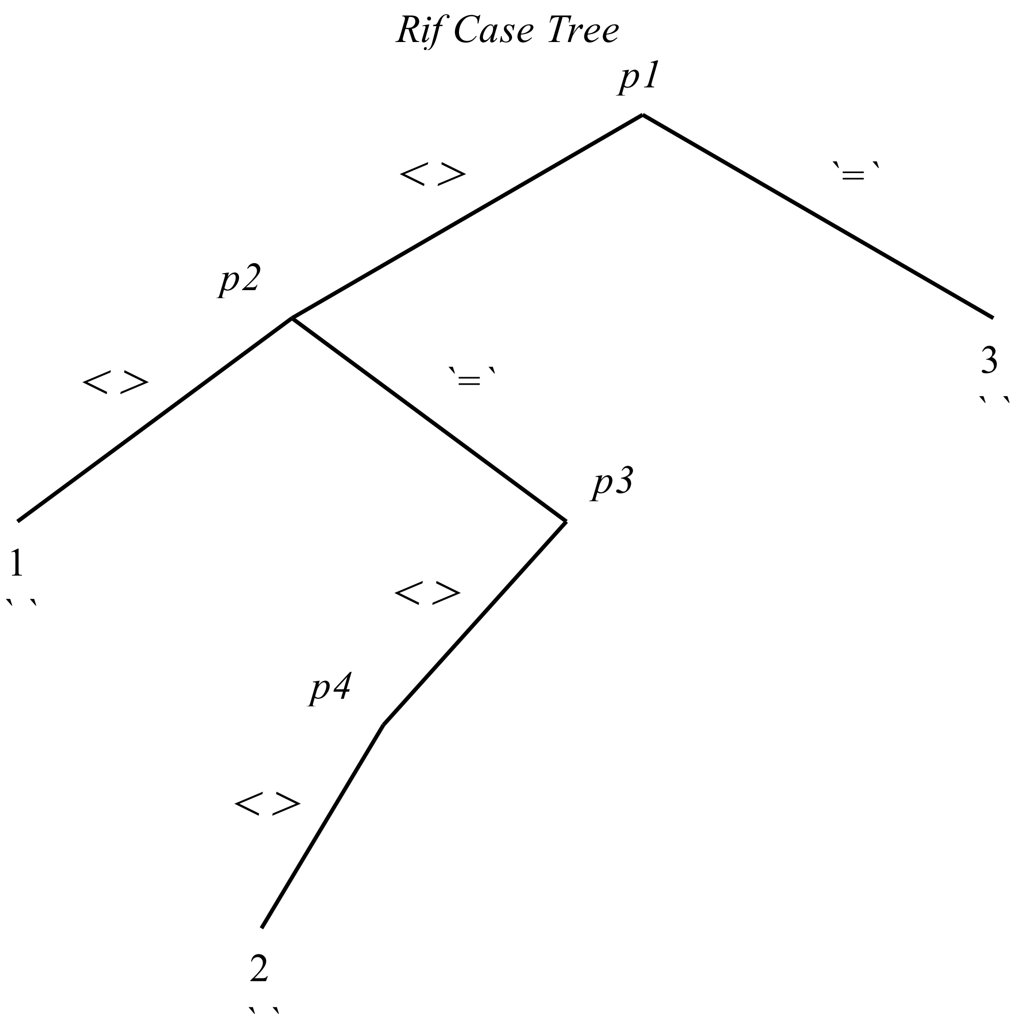
========= *Pivots Legend* =========

$$p1 = -r \Phi_r + 2 \Phi$$

$$p2 = r^2 \Phi \left(\nu_r r - 2 \right) \Phi_{r,r} - \left(r \Phi_r - 2 \Phi \right) \left(r^2 \nu_{r,r} \Phi + \left(-r^2 \nu_r + 2 r \right) \Phi_r + \Phi \left(\nu_r^2 r^2 - 2 \right) \right)$$

$$p3 = \Phi + (12 \epsilon + 12) r^2 \pi$$

$$p4 = -12 r \left(\frac{\Phi}{12} + \pi r^2 (\epsilon + 1) \right) \Phi_r + 2 \Phi^2 + 28 \pi r^2 (\epsilon + 1) \Phi + 32 \pi^2 r^4 (\epsilon + 1)^2$$



The third one, a singular case, is of reasonably small size ...

> *map*(*length*, *cases*)

$$[5399, 1661, 405] \tag{7.2.4.2}$$

An exact solution for it

> *sys*[3] := *op*(1, *cases*[3])

$$sys_3 := \left[e^{-\lambda} = -\frac{4\,\pi\,r^2\,(\epsilon+1)}{\Phi}, \lambda_r = 0, \nu_{r,r} = \frac{-r^4\,\pi\,(\epsilon+1)\,\nu_r^{\,2} + 2\,r^3\,\pi\,(\epsilon+1)\,\nu_r + \Phi + (4\,\epsilon+4)\,\pi\,r^2}{2\,r^4\,\pi\,(\epsilon+1)}, \Phi_r = \frac{2\,\Phi}{r} \right] \tag{7.2.4.3}$$

> *constraint*, *subsystem* := *selectremove*(*has*, *sys*[3], exp)

$$constraint, subsystem := \left[e^{-\lambda} = -\frac{4\,\pi\,r^2\,(\epsilon+1)}{\Phi} \right], \left[\lambda_r = 0, \nu_{r,r} = \frac{-r^4\,\pi\,(\epsilon+1)\,\nu_r^{\,2} + 2\,r^3\,\pi\,(\epsilon+1)\,\nu_r + \Phi + (4\,\epsilon+4)\,\pi\,r^2}{2\,r^4\,\pi\,(\epsilon+1)}, \Phi_r = \frac{2\,\Phi}{r} \right] \tag{7.2.4.4}$$

> $sol_{subsystem} := dsolve(subsystem, explicit)$

$$sol_{subsystem} := \left\{\Phi = \_C1\, r^2, \nu = -\frac{1}{\sqrt{\pi\,(\epsilon+1)}}\left(\ln\left(\frac{(32\,\epsilon+32)\,\pi+4\,\_C1}{\pi\,(\epsilon+1)\left(r^{\frac{\sqrt{(8\,\epsilon+8)\,\pi+\_C1}}{\sqrt{\pi\,(\epsilon+1)}}}\,\_C2-\_C3\right)^2}\right)\sqrt{\pi\,(\epsilon+1)}+\ln(r)\left(\sqrt{(8\,\epsilon+8)\,\pi+\_C1}-2\sqrt{\pi\,(\epsilon+1)}\right)\right), \lambda = \_C2\right\} \quad (7.2.4.5)$$

Specialize one of these constants using the constraint

> $eval(constraint, sol_{subsystem})$

$$\left[e^{-\_C2} = -\frac{4\,\pi\,(\epsilon+1)}{\_C1}\right] \quad (7.2.4.6)$$

> $solve(\textbf{(7.2.4.6)}, \_C1)$

$$\left\{\_C1 = -\frac{4\,\pi\,(\epsilon+1)}{e^{-\_C2}}\right\} \quad (7.2.4.7)$$

The exact solution

> $solution := subs(\textbf{(7.2.4.7)}, sol_{subsystem})$

$$solution := \left\{\Phi = -\frac{4\,\pi\,(\epsilon+1)\,r^2}{e^{-\_C2}}, \nu = -\frac{1}{\sqrt{\pi\,(\epsilon+1)}}\left(\ln\left(\frac{32\,e^{-\_C2}-16}{e^{-\_C2}\left(r^{\frac{\sqrt{4}\sqrt{\frac{\pi\,(\epsilon+1)\,(2\,e^{-\_C2}-1)}{e^{-\_C2}}}}{\sqrt{\pi\,(\epsilon+1)}}}\,\_C2-\_C3\right)^2}\right)\sqrt{\pi\,(\epsilon+1)}+\ln(r)\left(\sqrt{4}\sqrt{\frac{\pi\,(\epsilon+1)\,(2\,e^{-\_C2}-1)}{e^{-\_C2}}}\right.\right.\right. \quad (7.2.4.8)$$

$$-2\sqrt{\pi\,(\epsilon+1)}\Big)\Big), \lambda = \_C2\Big\}$$

Verifying this result

> $odetest(solution, ode_{system})$

$$\{0\} \tag{7.2.4.9}$$

>

*The Equivalence problem between two metrics

From the "What is new in Physics in Maple 2016" page:

In the Maple PDEtools package, you have the mathematical tools - including a complete symmetry approach - to work with the underlying [Einstein's] partial differential equations. [By combining that functionality with the one in the Physics and Physics:-Tetrads package] you can also formulate and, depending on the metrics also resolve, the equivalence problem; that is: to answer whether or not, given two metrics, they can be obtained from each other by a transformation of coordinates, as well as compute the transformation.

Example from: A. Karlhede, "A Review of the Geometrical Equivalence of Metrics in General Relativity", General Relativity and Gravitation, Vol. 12, No. 9, 1980

> $restart;$
> $with(Physics) : with(Tetrads) : Setup(auto = true, tetradmetric = null, signature = `+---`)$

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (see ?Physics,tetrads), $l_{\mu}, n_{\mu}, m_{\mu}, \overline{m}_{\mu}$

** Partial match of 'auto' against keyword 'automaticsimplification'*

$$[automaticsimplification = true, signature = + - - -, tetradmetric = \{(1,2) = 1, (3,4) = -1\}] \tag{7.3.1.1}$$

To formulate the problem, set first some symbols to represent the changed metric, changed mass and changed coordinates - no mathematics at this point

> $mt, tt, rt, thetat, phit := \mathfrak{m}, \mathfrak{t}, \mathfrak{r}, \vartheta, \varphi$

$$mt, tt, rt, thetat, phit := \mathfrak{m}, \mathfrak{t}, \mathfrak{r}, \vartheta, \varphi \tag{7.3.1.2}$$

Set now a new coordinates system, call it Y, involving the *new coordinates* (in the paper they are represented with a tilde on top of the letters)

> $Coordinates(Y = [tt, rt, thetat, phit])$

Default differentiation variables for d_, D_ and dAlembertian are: $\{Y = (t, r, \vartheta, \varphi)\}$

Systems of spacetime Coordinates are: $\{Y = (t, r, \vartheta, \varphi)\}$

$$\{Y\} \tag{7.3.1.3}$$

According to eq.(7.6) of the paper, the line element of Schwarzschild solution in isotropic spherical coordinates is given by

> $ds^2 := \left(\frac{\left(1 - \frac{mt}{2\,rt}\right)}{\left(1 + \frac{mt}{2\,rt}\right)}\right)^2 d\_(tt)^2 - \left(1 + \frac{mt}{2\,rt}\right)^4 \cdot \left(d\_(rt)^2 + rt^2 d\_(thetat)^2 + rt^2 \sin(thetat)^2 \cdot d\_(phit)^2\right)$

$$ds^2 := \frac{(-2\,r + m)^2\,(\partial(t))^2}{(2\,r + m)^2} - \frac{(2\,r + m)^4\,\left((\partial(r))^2 + r^2\,(\partial(\vartheta))^2 + r^2\sin(\vartheta)^2\,(\partial(\varphi))^2\right)}{16\,r^4} \tag{7.3.1.4}$$

Set this to be the metric

> $Setup\left(metric = ds^2\right):$

> $g\_[\,]$

$$g_{\mu,\nu} = \begin{bmatrix} \frac{(-2\,r + m)^2}{(2\,r + m)^2} & 0 & 0 & 0 \\ 0 & -\frac{(2\,r + m)^4}{16\,r^4} & 0 & 0 \\ 0 & 0 & -\frac{(2\,r + m)^4}{16\,r^2} & 0 \\ 0 & 0 & 0 & -\frac{(2\,r + m)^4 \sin(\vartheta)^2}{16\,r^2} \end{bmatrix} \tag{7.3.1.5}$$

In connection with the transformation used further below, compute now the Petrov type and the Weyl scalars for this metric, just to have an idea of what is behind this metric.

> $PetrovType(\,)$

$$\text{"D"} \tag{7.3.1.6}$$

> $Weyl[scalars]$

$$\psi_0 = 0,\ \psi_1 = 0,\ \psi_2 = -\frac{64\,r^3\,m}{(2\,r + m)^6},\ \psi_3 = 0,\ \psi_4 = 0 \tag{7.3.1.7}$$

We see that the Weyl scalars *are already in canonical form,* only $\Psi_2 \neq 0$ and the important thing: it depends on only one coordinate, r .

We want to see if this metric **(7.3.1.5)** is equivalent to Schwarzschild metric in standard spherical coordinates

> $g\_[sc]$

$$\textit{Systems of spacetime Coordinates are: } \{X = (t, r, \theta, \phi), Y = (t, \mathfrak{r}, \vartheta, \varphi)\}$$

$$\textit{Default differentiation variables for d\_, D\_ and dAlembertian are: } \{X = (t, r, \theta, \phi)\}$$

$$\textit{The Schwarzschild metric in coordinates } [t, r, \theta, \phi]$$

$$\textit{Parameters: } [m]$$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r - 2m}{r} & 0 & 0 & 0 \\ 0 & \frac{r}{-r + 2m} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin(\theta)^2 \end{bmatrix} \quad \textbf{(7.3.1.8)}$$

The equivalence we want to resolve is regarding an arbitrary relationship $\mathfrak{m}(m)$ between the masses used in **(7.3.1.5)** and **(7.3.1.8)** and a generic change of variables from X to Y

> $TR := \{\phi = \Phi(Y), r = R(Y), t = T(Y), \theta = \Theta(Y)\}$

$$TR := \{\phi = \Phi(Y), r = R(Y), t = T(Y), \theta = \Theta(Y)\} \quad \textbf{(7.3.1.9)}$$

> $CompactDisplay(TR)$

$$\Phi(t, \mathfrak{r}, \vartheta, \varphi) \textit{ will now be displayed as } \Phi$$

$$R(t, \mathfrak{r}, \vartheta, \varphi) \textit{ will now be displayed as } R$$

$$T(t, \mathfrak{r}, \vartheta, \varphi) \textit{ will now be displayed as } T$$

$$\Theta(t, \mathfrak{r}, \vartheta, \varphi) \textit{ will now be displayed as } \Theta \quad \textbf{(7.3.1.10)}$$

Using a differential equation mindset, ***the formulation*** of the equivalence between **(7.3.1.8)** and **(7.3.1.5)** under the transformation **(7.3.1.9)** is actually simple: change variables in **(7.3.1.8)**, using **(7.3.1.9)** and the Physics:-TransformCoordinates command (this is the command that changes variables in tensorial expressions), then equate the result to **(7.3.1.5)**, then try to solve the problem for the unknowns $\mathfrak{m}(m)$, $\Phi(Y)$, $R(Y)$, $\Theta(Y)$ and $T(Y)$.

We note at this point, however, that the Weyl scalars for Schwarzschild metric in this standard form **(7.3.1.8)** are also in canonical form of Petrov type D and also depend on only one variable, r

> $PetrovType(\,)$

$$\text{"D"} \quad \textbf{(7.3.1.11)}$$

> $Weyl[scalars]$

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = -\frac{m}{r^3}, \psi_3 = 0, \psi_4 = 0 \quad \textbf{(7.3.1.12)}$$

The fact that the Weyl scalars in both cases (**(7.3.1.7)** and **(7.3.1.12)**) are in canonical form (only $\Psi_2 \neq 0$) and in both cases this scalar depends on only one coordinate is already an indicator that the transformation involved changes only one variable in terms of the other one. So one could just search for a transformation of the form $r = R(\mathfrak{r})$ and resolve the problem instantly.

> $TransformCoordinates(r = R(rt), g\_[\mathrm{mu}, \mathrm{nu}])$

$$\begin{bmatrix} \frac{R(r)-2m}{R(r)} & 0 & 0 & 0 \\ 0 & \frac{R_r^{\,2} R(r)}{-R(r)+2m} & 0 & 0 \\ 0 & 0 & -R(r)^2 & 0 \\ 0 & 0 & 0 & -R(r)^2 \sin(\vartheta)^2 \end{bmatrix} \quad \textbf{(7.3.1.13)}$$

> *convert*(*rhs*(**(7.3.1.5)**) = **(7.3.1.13)**, *setofequations*)

$$\left\{0=0, \frac{(-2r+m)^2}{(2r+m)^2} = \frac{R(r)-2m}{R(r)}, -\frac{(2r+m)^4}{16r^4} = \frac{R_r^{\,2} R(r)}{-R(r)+2m}, -\frac{(2r+m)^4}{16r^2} = -R(r)^2, -\frac{(2r+m)^4 \sin(\vartheta)^2}{16r^2} = -R(r)^2\sin(\vartheta)^2\right\} \quad \textbf{(7.3.1.14)}$$

> *pdsolve*(**(7.3.1.14)**, [*R*, *mt*])

$$\left\{m=m, R(r) = \frac{(2r+m)^2}{4r}\right\}, \left\{m=-m, R(r) = -\frac{(m-2r)^2}{4r}\right\} \quad \textbf{(7.3.1.15)}$$

To make the problem slightly more general, consider instead a generic transformation for r in terms of all of $Y = (t, r, \vartheta, \varphi)$, and also allow the time to change, so we search for two transformation functions resolving the equivalence

> *tr* := *select*(*has*, *TR*, [*r*, *t*])

$$tr := \{r = R, t = T\} \quad \textbf{(7.3.1.16)}$$

> *CompactDisplay*(**(7.3.1.16)**)

$R(t, r, \vartheta, \varphi)$ *will now be displayed as* R

$T(t, r, \vartheta, \varphi)$ *will now be displayed as* T **(7.3.1.17)**

Transform the coordinates in the metric

> *TransformCoordinates*(*tr*, *g_*[mu, nu])

$$\left[\left[\frac{-4\left(-\frac{R}{2}+m\right)^2 T_t^{\,2} + R_t^{\,2} R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_r T_t + R_r R_t R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\vartheta T_t + R_\vartheta R_t R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_t + R_\varphi R_t R^2}{R(-R+2m)}\right], \right. \quad \textbf{(7.3.1.18)}$$

$$\left[\frac{-4\left(-\frac{R}{2}+m\right)^2 T_r T_t + R_r R_t R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_r^{\,2} + R_r^{\,2} R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\vartheta T_r + R_\vartheta R_r R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_r + R_\varphi R_r R^2}{R(-R+2m)}\right],$$

$$\left[\frac{-4\left(-\frac{R}{2}+m\right)^2 T_\vartheta T_t + R_\vartheta R_t R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\vartheta T_r + R_\vartheta R_r R^2}{R(-R+2m)},\right.$$

$$\left.\frac{T_\vartheta^2 (R-2m)}{R} - \frac{R_\vartheta^2 R}{R-2m} - R^2, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_\vartheta + R_\varphi R_\vartheta R^2}{R(-R+2m)}\right],$$

$$\left[\frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_t + R_\varphi R_t R^2}{R(-R+2m)}, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_r + R_\varphi R_r R^2}{R(-R+2m)},\right.$$

$$\frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_\vartheta + R_\varphi R_\vartheta R^2}{R(-R+2m)}, \frac{1}{R(-R+2m)}\left(-4\left(-\frac{R}{2}\right.\right.$$

$$\left.\left.\left.+m\right)^2 T_\varphi^2 + 2R^2\left(\frac{R_\varphi^2}{2} + R(\cos(\vartheta)-1)\left(-\frac{R}{2}+m\right)(\cos(\vartheta)+1)\right)\right)\right]\Bigg]$$

Change also the relationship between the masses so that $m(m) \neq m$, for instance:

> $subs\left(mt = \frac{1}{mt^2}, \mathbf{(7.3.1.5)}\right)$

$$g_{\mu,\nu} = \left[\left[\frac{\left(-2r+\frac{1}{m^2}\right)^2}{\left(2r+\frac{1}{m^2}\right)^2}, 0, 0, 0\right], \left[0, -\frac{\left(2r+\frac{1}{m^2}\right)^4}{16r^4}, 0, 0\right], \left[0, 0, -\frac{\left(2r+\frac{1}{m^2}\right)^4}{16r^2}, 0\right], \left[0, 0, 0, -\frac{\left(2r+\frac{1}{m^2}\right)^4 \sin(\vartheta)^2}{16r^2}\right]\right] \quad \mathbf{(7.3.1.19)}$$

> $convert(rhs(\mathbf{(7.3.1.19)}) = \mathbf{(7.3.1.18)}, setofequations)$

$$\left\{0 = \frac{-4\left(-\frac{R}{2}+m\right)^2 T_r T_t + R_r R_t R^2}{R\left(-R+2m\right)}, 0\right. \tag{7.3.1.20}$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_r + R_\varphi R_r R^2}{R\left(-R+2m\right)}, 0$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_t + R_\varphi R_t R^2}{R\left(-R+2m\right)}, 0$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\varphi T_\vartheta + R_\varphi R_\vartheta R^2}{R\left(-R+2m\right)}, 0$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\vartheta T_r + R_\vartheta R_r R^2}{R\left(-R+2m\right)}, 0$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_\vartheta T_t + R_\vartheta R_t R^2}{R\left(-R+2m\right)}, \frac{\left(-2\,r+\frac{1}{m^2}\right)^2}{\left(2\,r+\frac{1}{m^2}\right)^2}$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_t^{\,2} + R_t^{\,2} R^2}{R\left(-R+2m\right)}, -\frac{\left(2\,r+\frac{1}{m^2}\right)^4}{16\,r^4}$$

$$= \frac{-4\left(-\frac{R}{2}+m\right)^2 T_r^{\,2} + R_r^{\,2} R^2}{R\left(-R+2m\right)}, -\frac{\left(2\,r+\frac{1}{m^2}\right)^4}{16\,r^2} = \frac{T_\vartheta^{\,2}\left(R-2m\right)}{R}$$

$$-\frac{R_\vartheta^{\,2} R}{R-2m} - R^2, -\frac{\left(2\,r+\frac{1}{m^2}\right)^4 \sin(\vartheta)^2}{16\,r^2} = \frac{1}{R\left(-R+2m\right)}\Bigg($$

$$-4\left(-\frac{R}{2}+m\right)^2 T_\varphi^{\,2} + 2R^2\left(\frac{R_\varphi^{\,2}}{2} + R\left(\cos(\vartheta)-1\right)\left(-\frac{R}{2}+m\right)\left(\cos(\vartheta)\right)\right.$$

$$\left.\left.\left.+1\right)\right)\right)\Big\}$$

This problem, shown in Karlhede's paper as the example of the approach he summarized, is solvable using only differential elimination, in no time, obaining the same solution shown in the paper with equation number (7.10)

> *PDEtools*:-*casesplit*(**(7.3.1.20)**, [*R*, *T*, *mt*])

$$\left[R=-\frac{(m-2r)^2}{4r}, T_t=-1, T_r=0, T_\vartheta=0, T_\varphi=0, m^2=-\frac{1}{m}\right] \&\text{where } [m \neq 0], \left[R=-\frac{(m-2r)^2}{4r}, T_t=1, T_r=0, T_\vartheta=0, T_\varphi=0, m^2=-\frac{1}{m}\right] \&\text{where } [m \neq 0], \left[R=\frac{(2r+m)^2}{4r}, T_t=-1, T_r=0, T_\vartheta=0, T_\varphi=0, m^2=\frac{1}{m}\right] \&\text{where } [m \neq 0], \left[R=\frac{(2r+m)^2}{4r}, T_t=1, T_r=0, T_\vartheta=0, T_\varphi=0, m^2=\frac{1}{m}\right] \&\text{where } [m \neq 0] \quad \textbf{(7.3.1.21)}$$

> *pdsolve*(**(7.3.1.21)**[1], [*R*, *T*, *mt*])

$$\left\{m=-\frac{1}{\sqrt{-m}}, R=-\frac{(m-2r)^2}{4r}, T=-t+\_C2\right\}, \left\{m=\frac{1}{\sqrt{-m}}, R=-\frac{(m-2r)^2}{4r}, T=-t+\_C1\right\} \quad \textbf{(7.3.1.22)}$$

The fact that the time t appears defined in terms of the transformed time $\mathrm{T}(Y) = -t+\_C1$ involving an arbitrary constant is expected: the time does not enter the metric, it only enters through derivatives of $\mathrm{T}(Y)$ entering the Jacobian of the transformation used to change variables in tensorial expressions (the metric) in **(7.3.1.18)**.

Summary: the approach shown above, based on formulating the problem for the transformation functions of the equivalence and solving for them the differential equations using the commands in PDEtools, after restricting the generality of the transformation functions by looking at the form of the Weyl scalars, works well for other cases too, specially now that, in Maple 2016, the Weyl scalars can be expressed also in canonical form in one go (see previous Mapleprimes post on "Tetrads and Weyl scalars in canonical form"). Also important: in Maple 2016 it is present the functionality necessary to implement the approach of section 9.2 of the Exact solutions book as well.

>

**Equivalence for Schwarzschild metric (spherical and Krustal coordinates)*

This problem is interesting because:

a) It is well known in the literature
b) It involves departing from a metric expressed in "mixed coordinates"
c) When writting the metric entirely in Krustal coordinates, the dependence involves special

functions (LambertW)

Formulation of the problem (remove mixed coordinates)

> *restart*;

> *with*(*Physics*) : *with*(*Tetrads*) : *Setup*(*auto* = *true*);

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (see ?Physics,tetrads), $l_{\mu}, n_{\mu}, m_{\mu}, \overline{m}_{\mu}$

** Partial match of 'auto' against keyword 'automaticsimplification'*

$$[automaticsimplification = true] \quad (7.3.2.1.1)$$

The departure point, Schwarzschild metric in spherical coordinates

> g_[sc]

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu,\nu} = \begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \quad (7.3.2.1.2)$$

Introduce now Krustal coordinates following the literature (see wikipedia) and the corresponding line element involving "mixed" coordinates

> *Coordinates*($K = [u, \vartheta, \varphi, v]$)

Systems of spacetime Coordinates are: $\{K = (u, \vartheta, \varphi, v), X = (r, \theta, \phi, t)\}$

$$\{K, X\} \quad (7.3.2.1.3)$$

> $ds^2 := \frac{16\,(\partial(v))\, m^2\, e^{-\frac{r}{2m}}\,(\partial(u)) - \left(\left(1-\cos(\vartheta)^2\right)(\partial(\varphi))^2 + (\partial(\vartheta))^2\right) r^3}{r}$

$$ds^2 := \frac{1}{r}\left(16\,(\partial(v))\, m^2\, e^{-\frac{r}{2m}}\,(\partial(u)) - \left(\left(1-\cos(\vartheta)^2\right)(\partial(\varphi))^2 + (\partial(\vartheta))^2\right) r^3\right) \quad (7.3.2.1.4)$$

The mixing of variables is visible: in the line element above is in Krustal coordinates but you also see r , which belongs to the X (not K) coordinates.

For the purpose of formulating problem *free of this mixing of coordinates*, set the metric now to be **(7.3.2.1.4)**

> *Setup*(*diff* = [*K*], *metric* = **(7.3.2.1.4)**, *quiet*) :

> g_[ ]

$$g_{\mu,\nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{8\,m^2\,e^{-\frac{r}{2\,m}}}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin(\vartheta)^2 & 0 \\ \frac{8\,m^2\,e^{-\frac{r}{2\,m}}}{r} & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(7.3.2.1.5)}$$

To remove the mix of coordinates, introduce a transformation with unknown transformation functions $\{f, h\}$, change variables, and resolve for the transformation functions $\{f, g\}$ (this in itself is resolving a form of *equivalence* problem).

> $tr_0 := \{u = f(r,t), v = h(r,t)\}$

$$tr_0 := \{u = f(r,t), v = h(r,t)\} \qquad \textbf{(7.3.2.1.6)}$$

> *CompactDisplay*(tr_0)

f(r, t) will now be displayed as f

h(r, t) will now be displayed as h **(7.3.2.1.7)**

> *TransformCoordinates*(tr_0, g_[mu, nu])

$$\left[\left[\frac{16\,f_r\,h_r\,m^2\,e^{-\frac{r}{2\,m}}}{r}, 0, 0, \frac{8\,m^2\,e^{-\frac{r}{2\,m}}\left(f_r\,h_t + h_r\,f_t\right)}{r}\right], \left[0, -r^2, 0, 0\right], \left[0, 0, -r^2\sin(\theta)^2, 0\right], \left[\frac{8\,m^2\,e^{-\frac{r}{2\,m}}\left(f_r\,h_t + h_r\,f_t\right)}{r}, 0, 0, \frac{16\,f_t\,h_t\,m^2\,e^{-\frac{r}{2\,m}}}{r}\right]\right] \qquad \textbf{(7.3.2.1.8)}$$

Equate to **(7.3.2.1.2)** and solve

> *convert*(**(7.3.2.1.8)** = *rhs*(**(7.3.2.1.2)**), *setofequations*)

$$\left\{0=0, -r^2=-r^2, -r^2\sin(\theta)^2=-r^2\sin(\theta)^2, \frac{8 m^2 e^{-\frac{r}{2m}}\left(f_r h_t+h_r f_t\right)}{r}=0, \frac{16 f_r h_r m^2 e^{-\frac{r}{2m}}}{r}=\frac{r}{-r+2m}, \frac{16 f_t h_t m^2 e^{-\frac{r}{2m}}}{r}=\frac{r-2m}{r}\right\} \quad \textbf{(7.3.2.1.9)}$$

\> *pdsolve*(**(7.3.2.1.9)**)

$$\left\{f=\_C1+\_C2\sqrt{r-2m}\, e^{\frac{r}{4m}}\, e^{-\frac{t}{4m}}, h=-\frac{e^{\frac{r+t}{4m}}\sqrt{r-2m}}{\_C2}+\_C3\right\}, \quad \textbf{(7.3.2.1.10)}$$

$$\left\{f=\_C1+\_C2\sqrt{r-2m}\, e^{\frac{r}{4m}}\, e^{\frac{t}{4m}}, h=-\frac{\sqrt{r-2m}\, e^{\frac{r-t}{4m}}}{\_C2}+\_C3\right\}$$

Without loss of generality, set $[\_C1=0, \_C2=1, \_C3=0]$

\> $tr := combine(subs([\_C1=0, \_C2=1, \_C3=0], eval($**(7.3.2.1.6)**, **(7.3.2.1.10)**$[1])))$

$$tr := \left\{u=\sqrt{r-2m}\, e^{\frac{r-t}{4m}}, v=-e^{\frac{r+t}{4m}}\sqrt{r-2m}\right\} \quad \textbf{(7.3.2.1.11)}$$

Check it out:

\> g_[ ]

$$g_{\mu,\nu}=\begin{bmatrix} 0 & 0 & 0 & \frac{8 m^2 e^{-\frac{r}{2m}}}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin(\vartheta)^2 & 0 \\ \frac{8 m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(7.3.2.1.12)}$$

\> *TransformCoordinates*(*tr*, g_[mu, nu], [*X*], [*K*])

$$\begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \quad \textbf{(7.3.2.1.13)}$$

Here is where things become computationally challenguing: compute the inverse of the transformation **(7.3.2.1.11)**

> $itr := simplify(normal(solve(\mathbf{(7.3.2.1.11)}, \{r, t\}), expanded))$

$$itr := \left\{r = 2\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right) + 1\right) m,\ t = 2\ln\left(-\frac{v}{u}\right) m\right\} \qquad \mathbf{(7.3.2.1.14)}$$

This *itr* involves the LambertW function. Set now the metric to be the standard Schwarzschild's metric in spherical coordinates **(7.3.2.1.2)** and compute use *itr* to get he *form of the metric entirely in Krustal coordinates* - no more mixings

> $g\_[sc]$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu,\nu} = \begin{bmatrix} \frac{r}{-r+2\,m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2\,m}{r} \end{bmatrix} \qquad \mathbf{(7.3.2.1.15)}$$

So this is Schwarzschild's solution all in Krustal coordinates

> $TransformCoordinates(itr, g\_[\mathrm{mu}, \mathrm{nu}], [K], [X])$

$$\left[\left[0, 0, 0, -\frac{8\,W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right) m^2}{\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right) u\,v}\right], \left[0, -4\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)^2 m^2, 0, 0\right], \left[0, 0, -4\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)^2 m^2\sin(\vartheta)^2, 0\right], \left[-\frac{8\,W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right) m^2}{\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right) u\,v}, 0, 0, 0\right]\right] \qquad \mathbf{(7.3.2.1.16)}$$

This metric involves the LambertW function in a non-simplifiable form (to avoid that is the reason for people to use the mixed coordinates version **(7.3.2.1.5)**).

>

Solving the Equivalence

We now have the two forms: **(7.3.2.1.13)** in spherical and **(7.3.2.1.16)** in Krustal coordinates, so we can formulate the equivalence problem from one coordinate system to the other one.

The transformation to be resolved does not need to involve ϕ because neither ϕ nor φ enter either of the two metrics.
The transformation does not need to involve θ or ϑ because they enter the metrics in exactly the same position and with the same dependence.
In addition the Weyl scalars of both metrics are in canonical form and the only scalar different from zero, that is Ψ_2 does not depend on any of $\{\phi, \theta, \varphi, \vartheta\}$)

So we look for a generic transformation from spherical to Krustal of the form

$$\varphi$$
$$\vartheta \qquad \textbf{(7.3.2.2.1)}$$

> $\{r = R(K), t = T(K)\}$

$$\{r = R(K), t = T(K)\} \qquad \textbf{(7.3.2.2.2)}$$

> $CompactDisplay(\textbf{(7.3.2.2.2)})$

$$R(u, \vartheta, \varphi, v) \textit{ will now be displayed as } R$$
$$T(u, \vartheta, \varphi, v) \textit{ will now be displayed as } T \qquad \textbf{(7.3.2.2.3)}$$

The metric set in this moment is in spherical coordinates, **(7.3.2.1.15)**, so change using **(7.3.2.2.2)** and equate to **(7.3.2.1.16)** in Krustal coordinates

> $convert(TransformCoordinates(\textbf{(7.3.2.2.2)}, g\_[\mathrm{mu}, \mathrm{nu}], [K], [X]) = \textbf{(7.3.2.1.16)}, setofequations)$

$$\left\{\frac{-4\left(-\frac{R}{2}+m\right)^2 T_u^{\,2} + R_u^{\,2} R^2}{(-R+2m)R} = 0, \frac{-4\left(-\frac{R}{2}+m\right)^2 T_v^{\,2} + R_v^{\,2} R^2}{(-R+2m)R} = 0, \right. \qquad \textbf{(7.3.2.2.4)}$$

$$\frac{1}{(-R+2m)R}\left(-4\left(-\frac{R}{2}+m\right)^2 T_\varphi^{\,2} + 2R^2\left(\frac{R_\varphi^{\,2}}{2} + \left(-\frac{R}{2}\right.\right.\right.$$

$$\left.\left.\left.+m\right)R\,(\cos(\vartheta)-1)\,(\cos(\vartheta)+1)\right)\right) = -4\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2m}\right)\right.$$

$$\left.+1\right)^2 m^2 \sin(\vartheta)^2, \frac{-4\,T_v\left(-\frac{R}{2}+m\right)^2 T_u + R_v R_u R^2}{(-R+2m)R} =$$

$$- \frac{8\, W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right) m^2}{\left(W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right)+1\right) u\, v}, \frac{-4\, T_v \left(-\frac{R}{2}+m\right)^2 T_\varphi + R_v\, R_\varphi\, R^2}{(-R+2\, m)\, R} = 0,$$

$$\frac{-4\, T_v \left(-\frac{R}{2}+m\right)^2 T_\vartheta + R_v\, R_\vartheta\, R^2}{(-R+2\, m)\, R} = 0,$$

$$\frac{-4\, T_\varphi \left(-\frac{R}{2}+m\right)^2 T_u + R_\varphi\, R_u\, R^2}{(-R+2\, m)\, R} = 0,$$

$$\frac{-4\, T_\varphi \left(-\frac{R}{2}+m\right)^2 T_\vartheta + R_\varphi\, R_\vartheta\, R^2}{(-R+2\, m)\, R} = 0,$$

$$\frac{-4\, T_\vartheta \left(-\frac{R}{2}+m\right)^2 T_u + R_\vartheta\, R_u\, R^2}{(-R+2\, m)\, R} = 0, - \frac{{R_\vartheta}^2\, R}{R-2\, m} - R^2$$

$$+ \frac{{T_\vartheta}^2\, (R-2\, m)}{R} = -4 \left(W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right)+1\right)^2 m^2 \Bigg\}$$

Again, this is a nonlinear, non-rational PDE system in two unknowns depending on two independent variables (see **(7.3.2.2.2)**). You can now either call pdsolve on **(7.3.2.2.4)**, solving the problem in one step, or first split into cases without solving any differential equation, just doing differential elimination, to see the cases

> *PDEtools*:-*casesplit*(**(7.3.2.2.4)**)

$$\left[T_u = \frac{2\, m}{u}, T_\vartheta = 0, T_\varphi = 0, T_v = -\frac{2\, m}{v}, R = 2\left(W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right)+1\right) m\right] \quad \textbf{(7.3.2.2.5)}$$

$$\&\text{where } [\ ], \left[T_u = -\frac{2\, m}{u}, T_\vartheta = 0, T_\varphi = 0, T_v = \frac{2\, m}{v}, R = 2\left(W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right)+1\right) m\right] \&\text{where } [\ ]$$

So by only using differential elimination we removed all nonlinearities. This problem is actually easy for the differential equation routines

> *pdsolve*(**(7.3.2.2.4)**)

$$\left\{R = 2\left(W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right)+1\right) m, T = -2\, m \ln(u) + 2\, m \ln(v) + \_C1\right\}, \left\{R = 2\left(W\left(-\frac{u\, v\, e^{-1}}{2\, m}\right)+1\right) m, T = 2\, m \ln(u) - 2\, m \ln(v) + \_C1\right\} \quad \textbf{(7.3.2.2.6)}$$

So *the transformation of coordinates resolving the equivalence between* **(7.3.2.1.13)** *and* **(7.3.2.1.16)** *is*

> *eval*(**(7.3.2.2.2)**, **(7.3.2.2.6)**[1])

$$\left\{r = \left(2\, W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) + 2\right) m,\ t = -2\, m \ln(u) + 2\, m \ln(v) + \_C1\right\} \tag{7.3.2.2.7}$$

Check it transforming **(7.3.2.1.13)** fully written in spherical coordinates into **(7.3.2.1.16)** fully written in Krustal coordinates

> g_[ ]

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2\, m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r - 2\, m}{r} \end{bmatrix} \tag{7.3.2.2.8}$$

> *TransformCoordinates*(**(7.3.2.2.7)**, g_[mu, nu], [*K*], [*X*])

$$\left[\left[0, 0, 0, -\frac{8\, W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) m^2}{\left(W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) + 1\right) u\, v}\right], \left[0, -4 \left(W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) + 1\right)^2 m^2, 0, 0\right], \left[0, 0, -4 \left(W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) + 1\right)^2 m^2 \sin(\vartheta)^2, 0\right], \left[-\frac{8\, W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) m^2}{\left(W\left(-\frac{u\, v\, \mathrm{e}^{-1}}{2\, m}\right) + 1\right) u\, v}, 0, 0, 0\right]\right] \tag{7.3.2.2.9}$$

>

Tetrads and Weyl scalars in canonical form

Generally speaking a canonical form is obtained using transformations that leave invariant the tetrad metric in a tetrad system of references, so that theWeyl scalars are fixed as much as possible (conventionally, either equal to 0 or to 1).

Bringing a tetrad in canonical form is a relevant step in the tackling of the equivalence problem between two spacetime metrics.

The implementation is as in "General Relativity, an Einstein century survey", edited by S.W. Hawking (Cambridge) and W. Israel (U. Alberta, Canada), specifically Chapter 7 written by S. Chandrasekhar, page 388:

	Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Residual invariance
Petrov type I	0	$\neq 0$	$\neq 0$	1	0	none
Petrov type II	0	0	$\neq 0$	1	0	none
Petrov type III	0	0	0	1	0	none
Petrov type D	0	0	$\neq 0$	0	0	Ψ_2 remains invariant under rotations of Class III
Petrov type N	0	0	0	0	1	Ψ_4 remains invariant under rotations of Class II

The transformations (rotations of the tetrad system of references) used are of Class I, II and III as defined in Chandrasekar's chapter - equations (7.79) in page 384, (7.83) and (7.84) in page 385. Transformations of Class I can be performed with the command *Physics:-Tetrads:-TransformTetrad* using the optional argument *nullrotationwithfixedl_*, of Class II using *nullrotationwithfixedn_* and of Class III by calling *TransformTetrad*(*spatialrotationsm_mb_plan, boostsn_l_plane*), so with the two optional arguments simultaneously.

The determination of appropriate transformation parameters to be used in these rotations, as well as the sequence of transformations happens all automatically by using the optional argument, *canonicalform* of *TransformTetrad* .

```
> restart;
  with(Physics) :
  with(Tetrads);
```

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (*see ?Physics,tetrads*), $l_{\mu}, n_{\mu}, m_{\mu}, \overline{m}_{\mu}$

$$[IsTetrad, NullTetrad, OrthonormalTetrad, PetrovType, SimplifyTetrad, TransformTetrad, e\_, eta\_, gamma\_, l\_, lambda\_, m\_, mb\_, n\_] \tag{7.4.1}$$

>

Petrov type I

The numbers below used to enter the metric always refer to the equation number in the "Exact solutions to Einstein's field equations" textbook

> g_[[12, 21, 1]]

Systems of spacetime Coordinates are: $\{X = (t, x, y, \phi)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (t, x, y, \phi)\}$

The McLenaghan, Tariq (1975), Tupper (1976) metric in coordinates $[t, x, y, \phi]$

Parameters: $[a, k, \kappa 0]$

Comments: _k parametrizes the most general electromagnetic invariant with respect to the last 3 Killing vectors

Resetting the signature of spacetime from "+ - - -" to `- + + +` in order to match the signature in the database of metrics:

$$g_{\mu, \nu} = \begin{bmatrix} -1 & 0 & 0 & 2y \\ 0 & \frac{a^2}{x^2} & 0 & 0 \\ 0 & 0 & \frac{a^2}{x^2} & 0 \\ 2y & 0 & 0 & x^2 - 4y^2 \end{bmatrix} \tag{7.4.1.1}$$

The default tetrad computed by the Physics package routines

> e_[ ]

$$e_{a, \mu} = \begin{bmatrix} -1 & 0 & 0 & 2y \\ 0 & \frac{a}{|x|} & 0 & 0 \\ 0 & 0 & \frac{a}{|x|} & 0 \\ 0 & 0 & 0 & |x| \end{bmatrix} \tag{7.4.1.2}$$

The corresponding Weyl scalars

> Weyl[scalars]

$$\psi_0 = \frac{4\,\mathrm{I}\, x^3 |x|^3 - |x|^6 + |x|^4 x^2 + |x|^2 x^4 - x^6}{4 a^2 |x|^4 x^2}, \psi_1 = 0, \psi_2 = \tag{7.4.1.3}$$

$$- \frac{(x^2 + |x|^2)\,(x^4 + |x|^4)}{4\,a^2\,|x|^4\,x^2},\ \psi_3 = 0,\ \psi_4 = \frac{4\,I\,x^3\,|x|^3 - |x|^6 + |x|^4\,x^2 + |x|^2\,x^4 - x^6}{4\,a^2\,|x|^4\,x^2}$$

... there is abs around. Let's assume everything is positive to simplify the presentation of formulas

> *Assume*$(x > 0, y > 0, a > 0)$

$$\{a::(0, \infty)\},\ \{x::(0, \infty)\},\ \{y::(0, \infty)\} \qquad \textbf{(7.4.1.4)}$$

The scalars are now simpler, although still not in "canonical form" because $\Psi_4 \neq 0$ and $\Psi_3 \neq 1$.

> *Weyl*[*scalars*]

$$\psi_0 = \frac{I}{a^2},\ \psi_1 = 0,\ \psi_2 = -\frac{1}{a^2},\ \psi_3 = 0,\ \psi_4 = \frac{I}{a^2} \qquad \textbf{(7.4.1.5)}$$

The Petrov type

> *PetrovType*( )

"I" **(7.4.1.6)**

In this case the Weyl scalars are in canonical form when $\Psi_0 = 0$, $\Psi_4 = 0$ and $\Psi_3 = 1$.

> *TransformTetrad*(*canonicalform*)

$$\Bigg[\bigg[- \frac{\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,(2\sqrt{2} + \sqrt{5})}{2\,a^2},\ \frac{\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,(\sqrt{2} + \sqrt{5})}{2\,a\,x},\ \frac{1}{2\,a\,x}, \qquad \textbf{(7.4.1.7)}$$
$$\frac{1}{6\,a^2}\Big(\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,(2\sqrt{2} + \sqrt{5})\,\big(x\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{2} + \sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{5}\,x + 6\,y\big)\Big)\bigg],$$
$$\bigg[- \frac{\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,(5\sqrt{2} + 4\sqrt{5})\,a^2}{10},$$
$$- \frac{\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,a^3\,(5\sqrt{2} + 2\sqrt{5})}{10\,x},\ - \frac{\sqrt{2}\,\sqrt{5}\,a^3}{10\,x},$$
$$\frac{1}{30}\Big(\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,a^2\,(5\sqrt{2} + 4\sqrt{5})\,\big(x\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{2} + \sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{5}\,x + 6\,y\big)\Big)\bigg],$$
$$\bigg[- \frac{I\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{2}}{4} - \frac{I\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{5}}{5} + \frac{\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{2}}{4} + \frac{\sqrt{\sqrt{2}\,\sqrt{5} - 3}\,\sqrt{5}}{10},$$

$$-\frac{\sqrt{\sqrt{2}\sqrt{5}-3}\,\left(5\,\mathrm{I}\sqrt{2}+2\,\mathrm{I}\sqrt{5}+5\sqrt{2}+4\sqrt{5}\right)a}{20\,x},$$

$$-\frac{\left(-10\,\mathrm{I}+\mathrm{I}\sqrt{2}\sqrt{5}-3\sqrt{2}\sqrt{5}\right)a}{20\,x},\ \frac{1}{60}\Big(\big(10\,\mathrm{I}+\mathrm{I}\sqrt{2}\sqrt{5}$$

$$-3\sqrt{2}\sqrt{5}\big)\left(2\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{5}\,y+2\,y\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{2}+3\,x\right)\Big)\Big],$$

$$\Big[\frac{\mathrm{I}\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{2}}{4}+\frac{\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{2}}{4}+\frac{\mathrm{I}\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{5}}{5}$$

$$+\frac{\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{5}}{10},$$

$$\frac{\sqrt{\sqrt{2}\sqrt{5}-3}\,\left(5\,\mathrm{I}\sqrt{2}+2\,\mathrm{I}\sqrt{5}-5\sqrt{2}-4\sqrt{5}\right)a}{20\,x},$$

$$\frac{\left(-10\,\mathrm{I}+\mathrm{I}\sqrt{2}\sqrt{5}+3\sqrt{2}\sqrt{5}\right)a}{20\,x},\ -\frac{1}{60}\Big(\big(10\,\mathrm{I}+\mathrm{I}\sqrt{2}\sqrt{5}$$

$$+3\sqrt{2}\sqrt{5}\big)\left(2\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{5}\,y+2\,y\sqrt{\sqrt{2}\sqrt{5}-3}\,\sqrt{2}+3\,x\right)\Big)\Big]\Big]$$

Despite the fact that the result is a much more complicated tetrad, this is an amazing result in that the resulting Weyl scalars are all fixed (see below). Let's first verify that this is indeed a tetrad, and that now the Weyl scalars are in canonical form

> *IsTetrad*(**(7.4.1.7)**)

Type of tetrad: null

$$true \qquad \textbf{(7.4.1.8)}$$

Set **(7.4.1.7)** to be the tetrad in use and recompute the Weyl scalars

> *Setup*(*tetrad* = **(7.4.1.7)**) :

Inded we now have $\Psi_0 = 0$, $\Psi_4 = 0$ and $\Psi_3 = 1$

> *simplify*([*Weyl*[*scalars*]])

$$\left[\psi_0 = 0,\ \psi_1 = \frac{-\frac{1}{2}-\frac{3\,\mathrm{I}}{2}}{a^4},\ \psi_2 = \frac{-1+\mathrm{I}}{a^2},\ \psi_3 = 1,\ \psi_4 = 0\right] \qquad \textbf{(7.4.1.9)}$$

So Weyl scalars computed after setting the canonical tetrad **(7.4.1.7)** to be the tetrad in use are in canonical form. Great! NOTE: computing the canonicalWeyl scalars is not really the difficult part, and within the code, these scalars **(7.4.1.9)** are computed before arriving at the tetrad **(7.4.1.7)**. What is really difficult (from the point of view of computational complexity and simplifications) is to compute the actual canonical form of the tetrad **(7.4.1.7)**.

>

Petrov type II

Consider this other solution to Einstein's equation (again, the numbers in g_[[24,37,7]] always refer to the equation number in the "Exact solutions to Einstein's field equations" textbook)

> g_[[24, 37, 7]]

Systems of spacetime Coordinates are: $\{X = (u, v, x, y)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (u, v, x, y)\}$

The Stephani metric in coordinates $[u, v, x, y]$

Parameters: $[f(x), a, \Psi 1(u, x, y)]$

Comments: Case 6 from Table 24.1:_Psi1(u,x,y): diff(_Psi1(u,x,y),x,x)+diff(_Psi1(u,x,y),y,y)=0, diff(x*diff(_M(u,x,y),x),x)+x*diff(_M(u,x,y),y,y)=_kappa0*(diff(_Psi(u,x,y),x)^2+diff(_Psi(u,x,y),y)^2)

$$g_{\mu,\nu} = \begin{bmatrix} -2x(f(x)+ya) & -x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{x}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{x}} \end{bmatrix} \quad \textbf{(7.4.2.1)}$$

Check the Petrov type

> *PetrovType*( )

"II" **(7.4.2.2)**

The starting tetrad

> e_[ ]

$$e_{a,\mu} = \begin{bmatrix} -\sqrt{x}\sqrt{f(x)+ya} & 0 & 0 & 0 \\ -\sqrt{x}\sqrt{f(x)+ya} & -\frac{\sqrt{x}}{\sqrt{f(x)+ya}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2x^{1/4}} & \frac{\frac{I}{2}\sqrt{2}}{x^{1/4}} \\ 0 & 0 & \frac{\sqrt{2}}{2x^{1/4}} & \frac{-\frac{I}{2}\sqrt{2}}{x^{1/4}} \end{bmatrix} \quad \textbf{(7.4.2.3)}$$

results in Weyl scalars not in canonical form:

> *Weyl*[*scalars*]

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = \frac{1}{8x^{3/2}}, \psi_3 = 0, \psi_4 = -\frac{3\,I\,a - 2x f''(x) - 3f'(x)}{\sqrt{x}\,(4ya + 4f(x))} \quad \textbf{(7.4.2.4)}$$

For Petrov type "II", the canonical form is as for type "I" but in addition $\Psi_1 = 0$. Again let's assume positive, not necessary, but to get simpler formulas around

> *Assume*$(f(x) > 0, x > 0, y > 0, a > 0)$

$\{a::(0, \infty]\}, \{x::(0, \infty], -f(x)::[-\infty, 0), f(x)::(0, \infty]\}, \{y::(0, \infty]\}$ **(7.4.2.5)**

Compute now a canonical form for the tetrad, to be used instead of **(7.4.2.3)**

> $TransformTetrad(canonicalform)$

$$\left[\left[-\frac{\sqrt{3}\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}}{8\,\sqrt{x}},\,0,\,0,\,0\right],\right. \tag{7.4.2.6}$$

$$\left[-\frac{8\,\sqrt{3}\,x^{3/2}\,\left(3\,x\,a+2\,I\,f''(x)\,x^2+3\,I\,f'(x)\,x+3\,y\,a+3\,f(x)\right)}{9\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}},\right.$$

$$-\frac{8\,\sqrt{3}\,x^{3/2}}{3\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}},$$

$$-\frac{4\,x^{5/4}\,\sqrt{2}\,(2\,I\,x\,f''(x)+3\,I\,f'(x)+3\,I\,a-2\,x\,f''(x)-3\,f'(x)+3\,a)}{3\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,\sqrt{3\,I\,a-2\,x\,f''(x)-3\,f'(x)}},$$

$$\left.\frac{4\,x^{5/4}\,\sqrt{2}\,(2\,I\,x\,f''(x)+3\,I\,f'(x)+3\,I\,a-2\,x\,f''(x)-3\,f'(x)+3\,a)}{3\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,\sqrt{3\,I\,a-2\,x\,f''(x)-3\,f'(x)}}\right],$$

$$\left[-\frac{\sqrt{3}\,x\,(2\,I\,x\,f''(x)+3\,I\,f'(x)+3\,a)}{3\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}},\,0,\right.$$

$$-\frac{\sqrt{3\,I\,a-2\,x\,f''(x)-3\,f'(x)}\,\sqrt{2}}{2\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,x^{1/4}},$$

$$\left.\frac{-\frac{I}{2}\,\sqrt{3\,I\,a-2\,x\,f''(x)-3\,f'(x)}\,\sqrt{2}}{\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,x^{1/4}}\right],$$

$$\left[-\frac{\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,\sqrt{3}\,x}{3},\,0,\right.$$

$$-\frac{\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,\sqrt{2}}{2\,\sqrt{3\,I\,a-2\,x\,f''(x)-3\,f'(x)}\,x^{1/4}},$$

$$\left.\left.\frac{\frac{I}{2}\,\sqrt{-I\,(3\,I\,a-2\,x\,f''(x)-3\,f'(x))}\,\sqrt{2}}{\sqrt{3\,I\,a-2\,x\,f''(x)-3\,f'(x)}\,x^{1/4}}\right]\right]$$

Set this tetrad and check the Weyl scalars again

> $Setup(tetrad = \mathbf{(7.4.2.6)})$:

> $Weyl[scalars]$

$$\psi_0 = 0,\ \psi_1 = 0,\ \psi_2 = \frac{1}{8\,x^{3/2}},\ \psi_3 = 1,\ \psi_4 = 0 \tag{7.4.2.7}$$

This result **(7.4.2.7)** is fantastic. Compare these Weyl scalars with the ones **(7.4.2.4)** before transforming the tetrad.

>

Petrov type III

> g_[[12, 35, 1]]

Systems of spacetime Coordinates are: $\{X=(u,x,y,z)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X=(u,x,y,z)\}$

The Kaigorodov (1962), Cahen (1964), Siklos (1981), Ozsvath (1987) metric in coordinates $[u,x,y,z]$

Parameters: $[\Lambda]$

$$g_{\mu,\nu}=\begin{bmatrix}0 & e^{-2z} & 0 & 0\\ e^{-2z} & e^{4z} & 2e^{z} & 0\\ 0 & 2e^{z} & 2e^{-2z} & 0\\ 0 & 0 & 0 & \frac{3}{|\Lambda|}\end{bmatrix} \quad (7.4.3.1)$$

> *Assume*(z > 0, Lambda > 0)

$$\{\Lambda::(0,\infty]\},\{z::(0,\infty]\} \quad (7.4.3.2)$$

The Petrov type and the original tetrad

> *PetrovType*( )

$$\text{"III"} \quad (7.4.3.3)$$

> e_[ ]

$$e_{a,\mu}=\begin{bmatrix}-\frac{I}{2}e^{-4z}(\sqrt{2}-2) & -\frac{I}{2}\sqrt{2}\,e^{2z} & -I\,e^{-z}(\sqrt{2}-1) & 0\\ -\frac{I}{2}e^{-4z}(2+\sqrt{2}) & -\frac{I}{2}\sqrt{2}\,e^{2z} & -I\,e^{-z}(1+\sqrt{2}) & 0\\ \frac{I}{2}\sqrt{2}\,e^{-4z} & 0 & 0 & \frac{\sqrt{2}\sqrt{3}}{2\sqrt{\Lambda}}\\ -\frac{I}{2}\sqrt{2}\,e^{-4z} & 0 & 0 & \frac{\sqrt{2}\sqrt{3}}{2\sqrt{\Lambda}}\end{bmatrix} \quad (7.4.3.4)$$

This tetrad results in the following scalars

> *Weyl*[*scalars*]

$$\psi_0=-2\Lambda\sqrt{2}+\frac{11\Lambda}{4},\ \psi_1=-\frac{\Lambda\sqrt{2}}{2}+\frac{3\Lambda}{4},\ \psi_2=\frac{\Lambda}{4},\ \psi_3=-\frac{\Lambda\sqrt{2}}{2}-\frac{3\Lambda}{4},\ \psi_4=2\Lambda\sqrt{2}+\frac{11\Lambda}{4} \quad (7.4.3.5)$$

that are not in canonical form, which for Petrov type III is as in Petrov type II but in addition we should have $\Psi_2=0$.

Compute now a canonical form for the tetrad

> *TransformTetrad*(*canonicalform*)

$$\begin{bmatrix} 0 & -\frac{I}{2}\,e^{2z}\,\Lambda\sqrt{2} & -\frac{I}{2}\,\Lambda\,e^{-z}\sqrt{2} & \frac{\sqrt{3}\sqrt{\Lambda}}{2} \\ \frac{\frac{I}{2}\,e^{-4z}\sqrt{2}}{\Lambda} & \frac{-\frac{13\,I}{8}\,e^{2z}\sqrt{2}}{\Lambda} & \frac{-\frac{9\,I}{8}\,e^{-z}\sqrt{2}}{\Lambda} & -\frac{7\sqrt{3}}{8\,\Lambda^{3/2}} \\ 0 & \frac{3\,I}{4}\,e^{2z}\sqrt{2} & \frac{I}{4}\,e^{-z}\sqrt{2} & -\frac{\sqrt{3}}{4\sqrt{\Lambda}} \\ -I\,e^{-4z}\sqrt{2} & \frac{3\,I}{4}\,e^{2z}\sqrt{2} & \frac{I}{4}\,e^{-z}\sqrt{2} & -\frac{\sqrt{3}}{4\sqrt{\Lambda}} \end{bmatrix} \quad \textbf{(7.4.3.6)}$$

Set this one to be the tetrad in use and recompute the Weyl scalars

> *Setup*(*tetrad* = **(7.4.3.6)**) :

> *Weyl*[*scalars*]

$$\psi_0 = 0,\ \psi_1 = 0,\ \psi_2 = 0,\ \psi_3 = 1,\ \psi_4 = 0 \quad \textbf{(7.4.3.7)}$$

>

Petrov type N

> g_[[12, 6, 1]]

Systems of spacetime Coordinates are: $\{X = (u, v, y, z)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (u, v, y, z)\}$

The Defrise (1969) metric in coordinates $[u, v, y, z]$

Parameters: $[\Lambda, \kappa 0]$

Comments: _Lambda < 0 required for a pure radiation solution

$$g_{\mu,\nu} = \begin{bmatrix} 0 & -\frac{3}{2\,y^2\,\Lambda} & 0 & 0 \\ -\frac{3}{2\,y^2\,\Lambda} & -\frac{3}{y^4\,\Lambda} & 0 & 0 \\ 0 & 0 & \frac{3}{y^2\,\Lambda} & 0 \\ 0 & 0 & 0 & \frac{3}{y^2\,\Lambda} \end{bmatrix} \quad \textbf{(7.4.4.1)}$$

> *Assume*($y > 0$, Lambda > 0)

$$\{\Lambda::(0, \infty]\},\ \{y::(0, \infty]\} \quad \textbf{(7.4.4.2)}$$

> *PetrovType*( )

"N" **(7.4.4.3)**

The original tetrad and related Weyl scalars are not in canonical form:

> $e\_[\,]$

$$e_{a,\mu} = \begin{bmatrix} -\frac{\sqrt{2}\sqrt{3}}{4\sqrt{\Lambda}} & -\frac{\sqrt{2}\sqrt{3}}{2y^2\sqrt{\Lambda}} & \frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} & 0 \\ -\frac{\sqrt{2}\sqrt{3}}{4\sqrt{\Lambda}} & -\frac{\sqrt{2}\sqrt{3}}{2y^2\sqrt{\Lambda}} & -\frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} & 0 \\ \frac{\frac{I}{4}\sqrt{2}\sqrt{3}}{\sqrt{\Lambda}} & 0 & 0 & \frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} \\ \frac{-\frac{I}{4}\sqrt{2}\sqrt{3}}{\sqrt{\Lambda}} & 0 & 0 & \frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} \end{bmatrix} \quad \textbf{(7.4.4.4)}$$

> $Weyl[scalars]$

$$\psi_0 = -\frac{\Lambda}{4}, \psi_1 = -\frac{I}{4}\Lambda, \psi_2 = \frac{\Lambda}{4}, \psi_3 = \frac{I}{4}\Lambda, \psi_4 = -\frac{\Lambda}{4} \quad \textbf{(7.4.4.5)}$$

For Petrov type "N", the canonical form has $\Psi_4 \neq 0$ and all the other $\Psi_n = 0$.

Compute a canonical form, set it to be the tetrad in use and recompute the Weyl scalars

> $TransformTetrad(canonicalform)$

$$\begin{bmatrix} 0 & -\frac{\sqrt{2}\sqrt{3}}{2y^2} & 0 & 0 \\ -\frac{\sqrt{2}\sqrt{3}}{2\Lambda} & -\frac{\sqrt{2}\sqrt{3}}{\Lambda y^2} & -\frac{\sqrt{2}\sqrt{3}}{\Lambda y} & 0 \\ 0 & \frac{\sqrt{2}\sqrt{3}}{2y^2\sqrt{\Lambda}} & \frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} & \frac{\frac{I}{2}\sqrt{2}\sqrt{3}}{y\sqrt{\Lambda}} \\ 0 & \frac{\sqrt{2}\sqrt{3}}{2y^2\sqrt{\Lambda}} & \frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} & \frac{-\frac{I}{2}\sqrt{2}\sqrt{3}}{y\sqrt{\Lambda}} \end{bmatrix} \quad \textbf{(7.4.4.6)}$$

> $Setup(tetrad = \textbf{(7.4.4.6)})$:

> $Weyl[scalars]$

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = 0, \psi_3 = 0, \psi_4 = 1 \quad \textbf{(7.4.4.7)}$$

>

Petrov type D

> $g\_[[12, 8, 4]]$

Systems of spacetime Coordinates are: $\{X = (t, x, y, z)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (t, x, y, z)\}$

The metric in coordinates $[t, x, y, z]$

Parameters: $[A, B]$

Comments: k = 0, kprime = 1, not an Einstein metric

$$g_{\mu,\nu} = \begin{bmatrix} -B^2 \sin(z)^2 & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & A^2 x^2 & 0 \\ 0 & 0 & 0 & B^2 \end{bmatrix} \qquad \textbf{(7.4.5.1)}$$

> $Assume\left(A > 0, B > 0, x > 0, 0 \le z \le \frac{\text{Pi}}{4}\right)$

$$\{A::(0, \infty]\}, \{B::(0, \infty]\}, \{x::(0, \infty]\}, \left\{z::\left[0, \frac{\pi}{4}\right]\right\} \qquad \textbf{(7.4.5.2)}$$

> *PetrovType*( )

"D" **(7.4.5.3)**

The default tetrad and related Weyl scalars are not in canonical form, which for Petrov type "D" is with $\Psi_2 \neq 0$ and all the other $\Psi_n = 0$

> *e_*[ ]

$$e_{a,\mu} = \begin{bmatrix} \frac{\sqrt{2}\, B \sin(z)}{2} & \frac{\sqrt{2}\, A}{2} & 0 & 0 \\ \frac{\sqrt{2}\, B \sin(z)}{2} & -\frac{\sqrt{2}\, A}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}\, A\, x}{2} & \frac{I}{2}\sqrt{2}\, B \\ 0 & 0 & \frac{\sqrt{2}\, A\, x}{2} & -\frac{I}{2}\sqrt{2}\, B \end{bmatrix} \qquad \textbf{(7.4.5.4)}$$

> *Weyl*[*scalars*]

$$\psi_0 = \frac{1}{4 B^2}, \psi_1 = 0, \psi_2 = \frac{1}{12 B^2}, \psi_3 = 0, \psi_4 = \frac{1}{4 B^2} \qquad \textbf{(7.4.5.5)}$$

Transform the tetrad, set it and recompute the Weyl scalars

> *TransformTetrad*(*canonicalform*)

(7.4.5.6)

$$\left[\begin{array}{cccc} \sqrt{2}\,B\sin(z) & 0 & 0 & B\sqrt{2} \\ \frac{\sqrt{2}\,B\sin(z)}{4} & 0 & 0 & -\frac{B\sqrt{2}}{4} \\ 0 & -\frac{I}{2}\sqrt{2}\,A & \frac{\sqrt{2}\,A\,x}{2} & 0 \\ 0 & \frac{I}{2}\sqrt{2}\,A & \frac{\sqrt{2}\,A\,x}{2} & 0 \end{array}\right] \qquad \textbf{(7.4.5.6)}$$

> *Setup*(*tetrad* = **(7.4.5.6)**) :

> *Weyl*[*scalars*]

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = -\frac{1}{6\,B^2}, \psi_3 = 0, \psi_4 = 0 \qquad \textbf{(7.4.5.7)}$$

Again the expected canonical form of the Weyl scalars, and $\Psi_2 \neq 0$ remains invariant under transformations of Class III.

>