Quantum Mechanics: Schrödinger vs Heisenberg picture

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Within the Schrödinger picture of Quantum Mechanics, the time evolution of the state of a system, represented by a Ket $| \psi(t) \rangle$, is determined by Schrödinger's equation:

$$i \hbar \frac{d}{dt} | \psi_t \rangle = H | \psi_t \rangle$$

where H, the Hamiltonian, as well as the quantum operators O_S representing observable quantities, are all time-independent.

Within the Heisenberg picture, a Ket $| \psi \rangle$ representing the state of the system *does not evolve with time*, but the operators $O_H(t)$ representing observable quantities, and *through them* the Hamiltonian H, do.

Problem: Departing from Schrödinger's equation,

a) Show that the expected value of a physical observable in Schrödinger's and Heisenberg's representations is the same, i.e. that

$$\langle \psi_t | O_S | \psi_t \rangle = \langle \psi | O_H(t) | \psi \rangle$$

b) Show that the evolution equation of a given observable O_H in Heisenberg's picture, equivalent to Schrödinger's equation, is given by:

$$\dot{O}_H(t) = \frac{-i \left[O_H(t), H \right]_-}{\hbar}$$

where in the right-hand-side we see the commutator of O_H with the Hamiltonian of the system.

Solution: Let O_S and O_H respectively be operators representing one and the same observable quantity in Schrödinger's and Heisenberg's pictures, and H be the operator representing the Hamiltonian of a physical system. All of these operators are Hermitian. So we start by setting up the framework for this problem accordingly, including that the time t and Planck's constant are real. To automatically combine powers of the same base (happening frequently in what follows) we also set *combinepowersofsamebase* = *true*. The following input/output was obtained using the latest Physics update (Aug/31/2016) distributed on the

Maplesoft R&D Physics webpage.

> $with(Physics) : interface(imaginaryunit = i) :$

> $Setup\left(hermitianoperators = \{H, O_H, O_S\}, realobjects = \{t, \hbar\}, combinepowersofsamebase = true\right)$

$$\left[combinepowersofsamebase = true, hermitianoperators = \{H, O_H, O_S\}, realobjects = \{\hbar, t\}\right] \quad (1)$$

Let's consider Schrödinger's equation

> $i \cdot \hbar \cdot diff(Ket(\psi, t), t) = H\, Ket(\psi, t)$

$$i\,\hbar \frac{d}{dt} \left| \psi_t \right\rangle = H \left| \psi_t \right\rangle \quad (2)$$

Now, H is time-independent, so **(2)** can be formally solved: $\psi(t)$ is obtained from the solution $\psi(0)$ at time $t = 0$, as follows:

> $T := \exp\left(-\frac{i \cdot H \cdot t}{\hbar}\right)$

$$T := e^{\frac{-i\,t\,H}{\hbar}} \quad (3)$$

> $Ket(\psi, t) = T \cdot Ket(\psi, 0)$

$$\left| \psi_t \right\rangle = e^{\frac{-i\,t\,H}{\hbar}} \left| \psi_0 \right\rangle \quad (4)$$

To check that **(4)** is a solution of **(2)**, substitute it in **(2)**:

> $eval(\textbf{(2)}, \textbf{(4)})$

$$H\, e^{\frac{-i\,t\,H}{\hbar}} \left| \psi_0 \right\rangle = H\, e^{\frac{-i\,t\,H}{\hbar}} \left| \psi_0 \right\rangle \quad (5)$$

Next, to relate the Schrödinger and Heisenberg representations of an Hermitian operator O representing an observable physical quantity, recall that the value expected for this quantity at time t during a measurement is given by the mean value of the corresponding operator (i.e., bracketing it with the state of the system $\left| \psi_t \right\rangle$).

So let O_S be an observable in the Schrödinger picture: its mean value is obtained by bracketing the operator with equation **(4)**:

> $Dagger(\textbf{(4)}) \cdot O_S \cdot \textbf{(4)}$

$$\left\langle \psi_t \middle| O_S \middle| \psi_t \right\rangle = \left\langle \psi_0 \middle| e^{\frac{i\,t\,H}{\hbar}}\, O_S\, e^{\frac{-i\,t\,H}{\hbar}} \middle| \psi_0 \right\rangle \quad (6)$$

The composed operator within the bracket on the right-hand-side is the operator O in Heisenberg's picture, $O_H(t)$:

> $Dagger(T) \cdot \mathrm{O}_S \cdot T = \mathrm{O}_H(t)$

$$e^{\frac{i t H}{\hbar}} \mathrm{O}_S \, e^{\frac{-i t H}{\hbar}} = \mathrm{O}_H(t) \tag{7}$$

Analogously, inverting this equation,

> $T \cdot \mathbf{(7)} \cdot Dagger(T)$

$$\mathrm{O}_S = e^{\frac{-i t H}{\hbar}} \mathrm{O}_H(t) \, e^{\frac{i t H}{\hbar}} \tag{8}$$

As an aside to the problem, we note from these two equations, and since the operator $T = e^{\frac{-i t H}{\hbar}}$ is unitary (because H is Hermitian), that the switch between Schrödinger's and Heisenberg's pictures is accomplished through a unitary transformation.

Inserting now this value of O_S from **(8)** in the right-hand-side of **(6)**, we get the answer to item **a)**

> $lhs(\mathbf{(6)}) = eval(rhs(\mathbf{(6)}), \mathbf{(8)})$

$$\langle \psi_t | \mathrm{O}_S | \psi_t \rangle = \langle \psi_0 | \mathrm{O}_H(t) | \psi_0 \rangle \tag{9}$$

where, on the left-hand-side, the Ket representing the state of the system is evolving with time (Schrödinger's picture), while on the the right-hand-side the Ket ψ_0 is constant and it is $\mathrm{O}_H(t)$, the operator representing an observable physical quantity, that evolves with time (Heisenberg picture). As expected, both pictures result in the same expected value for the physical quantity represented by O.

To complete item b), the derivation of the evolution equation for $\mathrm{O}_H(t)$, we take the time derivative of the equation **(7)**:

> $diff((rhs = lhs)(\mathbf{(7)}), t)$

$$\dot{\mathrm{O}}_H(t) = \frac{i H e^{\frac{i t H}{\hbar}} \mathrm{O}_S \, e^{\frac{-i t H}{\hbar}}}{\hbar} - \frac{i e^{\frac{i t H}{\hbar}} \mathrm{O}_S H e^{\frac{-i t H}{\hbar}}}{\hbar} \tag{10}$$

To rewrite this equation in terms of the commutator $[\mathrm{O}_S, H]_-$, it suffices to re-order the product $H e^{\frac{i t H}{\hbar}}$ placing the exponential first:

> $Library\text{:-}SortProducts\left(\mathbf{(10)}, \left[e^{\frac{i t H}{\hbar}}, H\right], usecommutator\right)$

$$\dot{\mathrm{O}}_H(t) = \frac{i e^{\frac{i t H}{\hbar}} H \mathrm{O}_S \, e^{\frac{-i t H}{\hbar}}}{\hbar} - \frac{i e^{\frac{i t H}{\hbar}} \left(H \mathrm{O}_S + [\mathrm{O}_S, H]_-\right) e^{\frac{-i t H}{\hbar}}}{\hbar} \tag{11}$$

> *Normal*((**11**))

$$\dot{O}_H(t) = \frac{-\mathrm{i}\, e^{\frac{\mathrm{i}\, t\, H}{\hbar}} \left[O_S, H\right]_- e^{\frac{-\mathrm{i}\, t\, H}{\hbar}}}{\hbar} \tag{12}$$

Finally, to express the right-hand-side in terms of $\left[O_H(t), H\right]_-$ instead of $\left[O_S, H\right]_-$, we take the commutator of the equation **(8)** with the Hamiltonian

> *Commutator*((**8**), H)

$$\left[O_S, H\right]_- = e^{\frac{-\mathrm{i}\, t\, H}{\hbar}} \left[O_H(t), H\right]_- e^{\frac{\mathrm{i}\, t\, H}{\hbar}} \tag{13}$$

Combining these two expressions, we arrive at the expected result for **b)**, the evolution equation of a given observable O_H in Heisenberg's picture

> *eval*((**12**), (**13**))

$$\dot{O}_H(t) = \frac{-\mathrm{i}\left[O_H(t), H\right]_-}{\hbar} \tag{14}$$