Within the **Schrödinger picture** of Quantum Mechanics, the time evolution of the state of a system, represented by a Ket $| \psi(t) \rangle$, is determined by Schrödinger's equation:

$$i \hbar \frac{d}{dt} | \psi_t \rangle = H | \psi_t \rangle$$

where $H$, the Hamiltonian, as well as the quantum operators $O_S$ representing observable quantities, are all time-independent.

Within the **Heisenberg picture**, a Ket $| \psi \rangle$ representing the state of the system *does not evolve with time*, but the operators $O_H(t)$ representing observable quantities, and *through them* the Hamiltonian $H$, do.

**Problem:** Departing from Schrödinger's equation,

a) Show that the expected value of a physical observable in Schrödinger's and Heisenberg's representations is the same, i.e. that

$$\langle \psi_t | O_S | \psi_t \rangle = \langle \psi | O_H(t) | \psi \rangle$$

b) Show that the evolution equation of a given observable $O_H$ in Heisenberg's picture, equivalent to Schrödinger's equation, is given by:

$$\dot{O}_H(t) = -i \left[ O_H(t), H \right] / \hbar$$

where in the right-hand-side we see the commutator of $O_H$ with the Hamiltonian of the system.

**Solution:** Let $O_S$ and $O_H$ respectively be operators representing one and the same observable quantity in Schrödinger's and Heisenberg's pictures, and $H$ be the operator representing the Hamiltonian of a physical system. All of these operators are Hermitian. So we start by setting up the framework for this problem accordingly, including that the time $t$ and Planck’s constant are real. To automatically combine powers of the same base (happening frequently in what follows) we also set $combinepowersofsamebase = true$. The following input/output was obtained using the latest Physics update (Aug/31/2016) distributed on the
Let's consider Schrödinger's equation

\[ i \cdot \hbar \cdot \frac{d}{dt} |\psi_t\rangle = H |\psi_t\rangle \]  \hspace{1cm} (2)

Now, \( H \) is time-independent, so (2) can be formally solved: \( \psi(t) \) is obtained from the solution \( \psi(0) \) at time \( t = 0 \), as follows:

\[ T := \exp \left( - \frac{iHt}{\hbar} \right) \]

\[ T := e^{-\frac{iHt}{\hbar}} \]  \hspace{1cm} (3)

\[ Ket(\psi, t) = T \cdot Ket(\psi, 0) \]

\[ \left| \psi_t \right\rangle = e^{-\frac{iHt}{\hbar}} \left| \psi_0 \right\rangle \]  \hspace{1cm} (4)

To check that (4) is a solution of (2), substitute it in (2):

\[ eval((2), (4)) \]

\[ H e^{-\frac{iHt}{\hbar}} \left| \psi_0 \right\rangle = H e^{-\frac{iHt}{\hbar}} \left| \psi_0 \right\rangle \]  \hspace{1cm} (5)

Next, to relate the Schrödinger and Heisenberg representations of an Hermitian operator \( O \) representing an observable physical quantity, recall that the value expected for this quantity at time \( t \) during a measurement is given by the mean value of the corresponding operator (i.e., bracketing it with the state of the system \( \left| \psi_t \right\rangle \)).

So let \( O_S \) be an observable in the Schrödinger picture: its mean value is obtained by bracketing the operator with equation (4):

\[ Tagger((4)) \cdot O_S \cdot (4) \]

\[ \langle \psi_t \left| O_S \right| \psi_t \rangle = \langle \psi_0 \left| e^{\frac{itH}{\hbar}} O_S e^{-\frac{iHt}{\hbar}} \right| \psi_0 \rangle \]  \hspace{1cm} (6)

The composed operator within the bracket on the right-hand-side is the operator \( O \) in Heisenberg's picture, \( O_H(t) \):
\[ \text{Dagger}(T) \cdot O_S \cdot T = O_H(t) \]

\[ e^{\frac{iH}{\hbar}} O_S e^{\frac{-iH}{\hbar}} = O_H(t) \quad (7) \]

Analogously, inverting this equation,

\[ T \cdot (7) \cdot \text{Dagger}(T) \]

\[ O_S = e^{\frac{-iH}{\hbar}} O_H(t) e^{\frac{iH}{\hbar}} \quad (8) \]

As an aside to the problem, we note from these two equations, and since the operator \( T = e^{\frac{-iH}{\hbar}} \) is unitary (because \( H \) is Hermitian), that the switch between Schrödinger's and Heisenberg's pictures is accomplished through a unitary transformation.

Inserting now this value of \( O_S \) from (8) in the right-hand-side of (6), we get the answer to item a)

\[ \text{lhs}((6)) = \text{eval(rhs ((6)), (8))} \]

\[ \langle \psi_t | O_S | \psi_t \rangle = \langle \psi_0 | O_H(t) | \psi_0 \rangle \quad (9) \]

where, on the left-hand-side, the Ket representing the state of the system is evolving with time (Schrödinger's picture), while on the right-hand-side the Ket \( \psi_0 \) is constant and it is \( O_H(t) \), the operator representing an observable physical quantity, that evolves with time (Heisenberg picture). As expected, both pictures result in the same expected value for the physical quantity represented by \( O \).

To complete item b), the derivation of the evolution equation for \( O_H(t) \), we take the time derivative of the equation (7):

\[ \text{diff}((\text{rhs = lhs})((7)), t) \]

\[ \dot{O}_H(t) = i \frac{H}{\hbar} e^{\frac{iH}{\hbar}} O_S e^{\frac{-iH}{\hbar}} - \frac{i e^{\frac{-iH}{\hbar}} O_S H e^{\frac{iH}{\hbar}}}{\hbar} \quad (10) \]

To rewrite this equation in terms of the commutator \([O_S, H]_\_\), it suffices to re-order the product \( H e^{\frac{iH}{\hbar}} \) placing the exponential first:

\[ \text{Library:-SortProducts} \left( (10), [e^{\frac{iH}{\hbar}}, H], \text{usecommutator} \right) \]

\[ \dot{O}_H(t) = i \frac{e^{\frac{-iH}{\hbar}} H O_S e^{\frac{iH}{\hbar}}}{\hbar} - \frac{i e^{\frac{-iH}{\hbar}} (H O_S + [O_S, H]_\_) e^{\frac{iH}{\hbar}}}{\hbar} \quad (11) \]
\[ N_{O_H}(t) = \frac{-i e^{\frac{i t H}{\hbar}} [O_S, H]_- e^{\frac{-i t H}{\hbar}}}{\hbar} \]  

(12)

Finally, to express the right-hand-side in terms of \([O_H(t), H]_-\) instead of \([O_S, H]_-\), we take the commutator of the equation (8) with the Hamiltonian

\[ \text{Commutator (8), } H \]

\[ [O_S, H]_- = e^{\frac{-i t H}{\hbar}} [O_H(t), H]_- e^{\frac{i t H}{\hbar}} \]  

(13)

Combining these two expressions, we arrive at the expected result for \(b\), the evolution equation of a given observable \(O_H\) in Heisenberg's picture

\[ \text{eval((12), (13))} \]

\[ O_H(t) = \frac{-i [O_H(t), H]_-}{\hbar} \]  

(14)