The FunctionAdvisor: extending information on mathematical functions with computer algebra algorithms

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Abstract:
A shift in paradigm is happening, from: encoding information into a database, to: encoding essential blocks of information together with algorithms within a computer algebra system. Then, the information is not only searchable but can also be recreated in many different ways and actually used to compute. This talk focuses on this shift in paradigm over a real case example: the digitizing of information regarding mathematical functions as the FunctionAdvisor project of the Maple computer algebra system.

The FunctionAdvisor (basic)

> restart; FunctionAdvisor();
The usage is as follows:
> FunctionAdvisor( topic, function, ... );
where 'topic' indicates the subject on which advice is required, 'function' is the name of a Maple function, and '...' represents possible additional input depending on the 'topic' chosen. To list the possible topics:
> FunctionAdvisor( topics );
A short form usage,
> FunctionAdvisor( function );
with just the name of the function is also available and displays a summary of information about the function.

> FunctionAdvisor(topics)
The topics on which information is available are:

[DE, analytic_extension, asymptotic_expansion, branch_cuts, branch_points, 
calling_sequence, class_members, classify_function, definition, describe, 
differentiation_rule, function_classes, identities, integral_form, known_functions, 
periodicity, plot, relate, required_assumptions, series, singularities, special_values, 
specialize, sum_form, symmetries, synonyms, table]

> FunctionAdvisor( function, quiet)
[trig, trigh, arctrig, arctrigh, elementary, GAMMA_related, Psi_related, Kelvin, Airy, 
Hankel, Bessel_related, 0F1, orthogonal_polynomials, Ei_related, erf_related, Kummer, 
Whittaker, Cylinder, 1F1, Elliptic_related, Legendre, Chebyshev, 2F1, Lommel, 
Struve_related, hypergeometric, Jacobi_related, InverseJacobi_related, 
Elliptic_doubly_periodic, Weierstrass_related, Zeta_related, complex_components, 
piecewise_related, Other, Bell, Heun, Appell, trigall, arctrigall, integral_transforms]
The 14 functions in the "Bessel_related" class are:

\[ \text{AiryAi, AiryBi, BesselI, BesselJ, BesselK, BesselY, HankelH1, HankelH2, KelvinBei, KelvinBer, KelvinHei, KelvinHer, KelvinKei, KelvinKer} \]

\[ \text{FunctionAdvisor}\,(\text{describe, BesselK}) \]
BesselK = Modified Bessel function of the second kind

\[ \text{FunctionAdvisor}\,(\text{sum, tan}) \]
\[ \tan(z) = \sum_{k=1}^{\infty} \frac{B_{2k} (-1)^{k-1} z^{-1} + 2^{-k} (4^{-k} - 16^{-k})}{\Gamma(2-k+1)}, \quad |z| < \frac{\pi}{2} \]

\[ \text{FunctionAdvisor}\,(\text{integral, Beta}) \]
\[ B(x, y) = \int_{0}^{1} x^{-1} (1 - x)^{y-1} \, dx, \quad 0 < \Re(x) \land 0 < \Re(y) \]

More complicated relationships between mathematical functions, computed using Maple internal database and algorithms.

\[ \text{FunctionAdvisor}\,(\text{specialize, HermiteH, KummerU}) \]
\[ H_a(z) = 2^a U\left(-\frac{a}{2}, \frac{1}{2}, z^2\right), \quad 0 < \Re(z) \lor (\Re(z) = 0 \land 0 < \Im(z)) \]

\[ \text{FunctionAdvisor}\,(\text{DE, EllipticF(z, k)}) \]
\[ f(z, k) = F(z, k), \quad \frac{\partial^2}{\partial k^2} f(z, k) = \frac{(-1 - 3 k^4 z^2 + (z^2 + 3) k^2) \left(\frac{\partial}{\partial k} f(z, k)\right)}{k^2 z^2 + (1 - k^2 + k \eta)} \]
\[ + \frac{\left(\frac{\partial}{\partial z} f(z, k)\right) z^2}{(k^2 - k^2 \eta)^2 z^2 - k^2 + 1} + \frac{(-k^2 z^2 + 1) f(z, k)}{k^4 z^2 - k^2 z^2 + 1}, \quad \frac{\partial^2}{\partial k \partial z} f(z, k) = \]
\[ - \frac{\left(\frac{\partial}{\partial z} f(z, k)\right) z^2 k^2}{k^2 z^2 - 1}, \quad \frac{\partial^2}{\partial z^2} f(z, k) = \frac{(-2 k^2 z^3 + (k^2 + 1) z) \left(\frac{\partial}{\partial z} f(z, k)\right)}{1 + k^2 z^4 + (k^2 - 1) z^2 + } \]

The information returned by the FunctionAdvisor command can be used for further computations: verify the above

\[ \text{pdetest}(op(\%)) \]
\[ [0, 0, 0] \]

The relation between all elementary functions and the \( pFq \) hypergeometric function:

\[ \text{FunctionAdvisor}\,(\text{elementary, quiet}) \]
[ arccos, arccosh, arccot, arccoth, arcscc, arccsc, arcsec, arcsech, arcsin, arcsinh, arctan, arctanh, cos, cosh, cot, coth, cscc, csch, exp, ln, sec, sech, sin, sinh, tan, tanh ]

\[ \text{map2(FunctionAdvisor, relate, (1.10), hypergeom)} \]
\[
\text{arccos}(z) = \frac{\pi}{2} - z \, _2F_1\left( \frac{1}{2}; \frac{3}{2}; z^2 \right), \quad \text{arccosh}(z) = \frac{\sqrt{-(-1+z)^2}}{-2+2z} \\
\text{arccot}(z) = \frac{\pi}{2} - z \, _2F_1\left( \frac{1}{2}, 1; \frac{3}{2}; z^2 \right), \quad \text{arccoth}(z) = \frac{\pi \sqrt{-(z-1)^2} + 2z \, _2F_1\left( \frac{1}{2}, 1; \frac{3}{2}; z^2 \right)}{2z - 2} \\
\text{arcsec}(z) = \frac{\pi}{2} - z \, _2F_1\left( \frac{1}{2}; \frac{3}{2}; z^2 \right), \quad \text{arcsech}(z) = \frac{\sqrt{-\frac{(z-1)^2}{z^2}} + 2z \, _2F_1\left( \frac{1}{2}, 1; \frac{3}{2}; \frac{1}{z^2} \right)}{2z - 2} \\
\text{arcsin}(z) = z \, _2F_1\left( \frac{1}{2}, \frac{3}{2}; \frac{1}{z^2} \right), \quad \text{arcsinh}(z) = z \, _2F_1\left( \frac{1}{2}, \frac{3}{2}; -z^2 \right) \\
\text{arctan}(z) = z \, _2F_1\left( \frac{1}{2}, 1; \frac{3}{2}; -z^2 \right), \quad \text{arctanh}(z) = z \, _2F_1\left( \frac{1}{2}, 1; \frac{3}{2}; z^2 \right), \quad \cos(z) = _0F_1\left( \frac{1}{2}; -\frac{z^2}{4} \right), \quad \cosh(z) = _0F_1\left( \frac{1}{2}; \frac{3}{2}; z^2 \right)
\]
\[
\frac{z^2}{4}, \cot(z) = \frac{\text{cf}_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)}{z F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right)}, \csc(z) = \frac{\text{cf}_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)}{z F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right)},
\]

\[
\frac{1}{z F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right)}, \text{csch}(z) = \frac{1}{z F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right)}, e^z = e^{\text{cf}_0\left(\frac{1}{2}; z\right)}, \ln(z) = (z - 1)
\]

\[
_2F_1(1, 1; 2; -z + 1), \sec(z) = \frac{1}{0 F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)}, \text{sech}(z) = \frac{1}{0 F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)}, \sin(z)
\]

\[
_2F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right), \sinh(z) = z_0 F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right), \tan(z) = \frac{z_0 F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right)}{0 F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)},
\]

\[
\text{tanh}(z) = \frac{z_0 F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right)}{0 F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)}
\]

All the 'specializations' of the \text{arcsin} function

\[> \text{FunctionAdvisor}\left(\text{specialize, arcsin}\right)\]

\[
\begin{align*}
\text{arcsin}(z) &= z F_1\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 0, z^2\right), \text{with no restrictions on } (z), \\
\text{arcsin}(z) &= z F_2\left(\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 0, z^2\right), \text{with no restrictions on } (z), \\
\text{arcsin}(z) &= z F_3\left(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 0, z^2\right), \text{with no restrictions on } (z), \\
\text{arcsin}(z) &= z F_4\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 0, z^2\right), \text{with no restrictions on } (z), \\
\text{arcsin}(z) &= z HC\left(0, \frac{1}{2}, 0, 0, \frac{1}{4}, \frac{z^2}{z^2 - 1}\right), \text{with no restrictions on } (z), \\
\text{arcsin}(z) &= z HG\left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, z^2\right), \text{with no restrictions on } (z), \\
\text{arcsin}(z) &= \pi^{-1}(\text{arcsec}(z))_1, (z - 1), \text{R}(z)
\end{align*}
\]
\[
\arcsin(z) = \frac{z \pi P \left(\frac{1}{2}, -\frac{1}{2}\right) \left(-2z^2 + 1\right)}{2}, \text{ with no restrictions on } (z),
\]

\[
\arcsin(z) = \frac{z \sqrt{\pi} \left(-2z^2 + 2\right)^{1/4} P \left(\frac{1}{2} \frac{-1}{2}\right) \left(-2z^2 + 1\right)}{2 \left(-2z^2\right)^{1/4}}, \text{ with no restrictions on } (z),
\]

\[
\arcsin(z) = \frac{z G^{1,2}_{1,2} \left(\begin{array}{c}
\frac{1}{2}, \frac{1}{2} \\
-\frac{1}{2} \\
0, -\frac{1}{2}
\end{array}\right)}{2 \sqrt{\pi}}, \text{ with no restrictions on } (z),
\]

\[
\arcsin(z) = \frac{\arccosh(z) (z - 1)}{\sqrt{- (z - 1)^2}},
\]

\[
\arcsin(z) = \frac{\pi}{2} + 2 \arccot \left(\frac{z}{1 + \sqrt{-z^2 + 1}}\right),
\]

\[
\arcsin(z) = \frac{1}{1z + \sqrt{-z^2 + 1} + 1} \left(2 1 \sqrt{-z^2 + 1} + 2 1 - 2z\right) \arccoth \left(\frac{-1z}{1 + \sqrt{-z^2 + 1}}\right) + 1 \left(1 + \sqrt{-z^2 + 1}\right) \pi \sqrt{- \left(\frac{1z}{1 + \sqrt{-z^2 + 1}} + 1\right)^2}, \text{ with no restrictions on } (z),
\]

\[
\arcsin(z) = \arccsc \left(\frac{1}{z}\right), \text{ with no restrictions on } (z),
\]

\[
\arcsin(z) = \frac{\pi}{2} - \arccsc \left(\frac{1}{z}\right),
\]

\[
\arcsin(z) = \frac{\pi}{2} + \frac{\text{arcsech} \left(\frac{1}{z}\right) (-1 + z)}{\sqrt{- (\frac{1}{z} - 1)^2 z^2}},
\]

\[
\arcsin(z) = -I \arcsinh(Iz), \text{ with no restrictions on } (z),
\]
\[ \arcsin(z) = 2 \arctan \left( \frac{z}{1 + \sqrt{-z^2 + 1}} \right), \text{ with no restrictions on } (z) \], \[ \arcsin(z) = -2 \, \text{artanh} \left( \frac{1z}{1 + \sqrt{-z^2 + 1}} \right), \text{ with no restrictions on } (z) \], \[ \arcsin(z) = z \, _2F_1 \left( \frac{1}{2} \right), \frac{1}{2}; \frac{3}{2}; z^2 \right), \text{ with no restrictions on } (z) \], \[ \arcsin(z) = -1 \ln \left( 1 + \sqrt{-z^2 + 1} \right), \text{ with no restrictions on } (z) \]

Summarize about a function

> FunctionAdvisor (GAMMA)

GAMMA

- describe
- definition
- analytic extension
- classify function
- periodicity
- plot
- singularities
- branch points
- branch cuts
- special values
- identities
- sum form
- series
- asymptotic expansion
- integral form
- differentiation rule
- DE

General description

- The FunctionAdvisor command provides information on the following topics.
The FunctionAdvisor command provides information on the following mathematical functions.

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<td>erf</td>
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<td>exp</td>
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<td>FresnelC</td>
<td>Fresnelf</td>
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<td>GAMMA</td>
<td>GaussAGM</td>
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<tr>
<td>JacobiAM</td>
<td>JacobiCD</td>
<td>JacobiCN</td>
<td>JacobiCS</td>
</tr>
</tbody>
</table>
Like the conversion facility for mathematical functions, the FunctionAdvisor command also works with the concept of function classes and considers assumptions on the function parameters, if any. The FunctionAdvisor command provides information on the following function classes.

`0F1`  `1F1`  `2F1`  Airy
arctrig  arctrigh  Bessel_related  Chebyshev
Cylinder  Ei_related  elementary  Elliptic_doubly_periodic
Elliptic_related  erf_related  GAMMA_related  Hankel
hypergeometric  Jacobi_related  Kelvin  Kummer
Legendre  Lommel  orthogonal_polynomials  Other
WhittakerM  WhittakerW  Wrightomega  Zeta
The FunctionAdvisor command can be considered to be between a help and a computational special function facility. Due to the wide range of information this command can handle and in order to facilitate its use, it includes two distinctive features:

- If you call the FunctionAdvisor command without arguments, it returns information that you can follow until the appropriate information displays.
- If you call the FunctionAdvisor command with a topic or function misspelled, but a match exists, it returns the information with a warning message.

You do not have to remember the exact Maple name of each mathematical function or the FunctionAdvisor topic. However, to avoid these messages and all FunctionAdvisor verbosity, specify the optional argument quiet when calling the FunctionAdvisor command from another routine.

Beyond the concept of a database

"Mathematical functions, are defined by algebraic expressions. So consider algebraic expressions in general ..."

Formal power series for algebraic expressions

This can go in a database

> restart; sin(x) = convert(sin(x), FormalPowerSeries)

\[ \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \]  

(2.1.1)

What follows builds on top of things like the above, but cannot go into a database: they are generic algebraic expressions

> \( \sqrt{1 - \sqrt{1 - x}} \)

\[ \sqrt{1 - \sqrt{1 - x}} \]  

(2.1.2)

> % = convert(%, FormalPowerSeries)

\[ \sqrt{1 - \sqrt{1 - x}} = \sum_{k=0}^{\infty} \frac{\sqrt{2} (4k)! 16^{-k} x^k}{2 (2k)!^2 (2k+1)} \]  

(2.1.3)

> e^\text{arcsin}(x)

\[ e^{\text{arcsin}(x)} \]  

(2.1.4)

> % = convert(%, FormalPowerSeries)
\[
e^{\arcsin(x)} = \left( \sum_{k=0}^{\infty} \frac{k^4}{4^k + 1} \frac{x^2}{2k+1} \right) + \left( \sum_{k=0}^{\infty} \frac{k^2 + k + 1}{2k^2 + 2k + 1} \frac{x^2}{2k+1} \right)
\]

\[
> \frac{t}{(1 - x t - t^2)}
\]

\[
> \frac{t}{-t^2 - x t + 1}
\]

\[
> \% = \text{convert(\%}, \text{FormalPowerSeries}, t)
\]

\[
\frac{t}{-t^2 - x t + 1} = \sum_{k=0}^{\infty} \left( -\left( \frac{x}{2} - \frac{\sqrt{x^2 + 4}}{2} \right)^k + \left( \frac{x}{2} + \frac{\sqrt{x^2 + 4}}{2} \right)^k \right)^k
\]

\[
> \ln \left( \frac{1 + x^2}{1 - x} \right)
\]

\[
> \% = \text{convert(\%}, \text{FormalPowerSeries, method = rational})
\]

\[
\ln \left( \frac{x^2 + 1}{1 - x} \right) = \sum_{k=0}^{\infty} \frac{(1 (-1)^k - 1 1^k + (-1)^k 1^k) x^k}{(-1)^k 1^k (k + 1)}
\]

\[
> e^x \sin(x)
\]

\[
> \% = \text{convert(\%}, \text{FormalPowerSeries, method = hypergeometric})
\]

\[
e^x \sin(x) = \sum_{k=0}^{\infty} \left( -\frac{1 (1 + 1)^k}{2k!} + \frac{1 (1 - 1)^k}{2k!} \right) x^k
\]

\[
> e^x - 2 e^{-\frac{x}{2}} \cos \left( \frac{\sqrt{3} x}{2} + \frac{\pi}{3} \right)
\]

\[
> \% = \text{convert(\%}, \text{FormalPowerSeries, method = exponential})
\]

\[
e^x - 2 e^{-\frac{x}{2}} \cos \left( \frac{\sqrt{3} x}{2} + \frac{\pi}{3} \right) = \sum_{k=0}^{\infty} \left( -\frac{\cos \left( \frac{2k \pi}{3} \right)}{k!} + \frac{\sqrt{3} \sin \left( \frac{2k \pi}{3} \right)}{k!} \right) x^k + \frac{1}{k!} x^k
\]

\[
> \%\% = \text{convert(\%\%}, \text{FormalPowerSeries, method = hypergeometric})
\]
\[ e^x - 2 e^{-\frac{x}{2}} \cos \left( \frac{\sqrt{3} x}{2} + \frac{\pi}{3} \right) = \sum_{k=0}^{\infty} \frac{3 x^{3k+1}}{(3k+1)!} \] (2.1.14)

**References**

**Differential polynomial forms for algebraic expressions**

**Extending the mathematical language to include the inverse functions**
Consider the inverse of the functions of the mathematical language, for example:

```plaintext
> with(PDEtools): declare(y(x), prime=x);

\( y(x) \) will now be displayed as \( y \)

\( \text{derivatives with respect to } x \text{ of functions of one variable will now be displayed with ' } \) (2.2.1.1)

> y(x) = dawson( -1 ) (x)

\( y = \text{dawson}^{-1}(x) \) (2.2.1.2)

> dpolyform ([2.2.1.2], no_Fn)

\[ y' = -\frac{1}{2 y x - 1} \] &where \([y \neq 0] \) (2.2.1.3)

Representing *inverse functions* in differential polynomial form is a starting point to transform all these *inverse functions* into actual functions extending the expressiveness of the mathematical language.

> op([1, 1 ], %), y(0) = alpha

\( y' = -\frac{1}{2 y x - 1}, y(0) = \alpha \) (2.2.1.4)

> dsolve([%], y(x), series)

\[ y = \alpha + x + \alpha x^2 + \left( \frac{4 \alpha^2}{3} + \frac{2}{3} \right) x^3 + \left( 2 \alpha^3 + \frac{5 \alpha}{2} \right) x^4 + \left( \frac{104}{15} \alpha^2 + \frac{16}{15} \right) x^5 + O(x^6) \] (2.2.1.5)

> y(x) = (erf@@(-1)) (x)

\( y = \text{erf}^{-1}(x) \) (2.2.1.6)

> dpolyform ([2.2.1.6], no_Fn)

\[ y'' = 2 y y'^2 \] &where \([y'' \neq 0] \) (2.2.1.7)

> op([1, 1 ], %), y(0) = alpha, D(y)(0) = beta

\( y'' = 2 y y'^2, y(0) = \alpha, D(y)(0) = \beta \) (2.2.1.8)

> dsolve([%], y(x), series)

\[ y = \alpha + \beta x + \alpha \beta^2 x^2 + \left( \frac{4}{3} \alpha^2 \beta^3 + \frac{1}{3} \beta^3 \right) x^3 + \left( \frac{7}{6} \beta^4 \alpha + 2 \alpha^3 \beta^4 \right) x^4 \] (2.2.1.9)
The Differential Equations representing arbitrary algebraic expressions

For ease of reading, these functions are declared to be displayed in a compact way. Also, derivatives are displayed as indexed objects.

Consider the following non-polynomial expression.

\[ f(x) = \tan(x) \]

\( f = \tan(x) \)  \hspace{1cm} (2.2.2.2)

\[ g(x, y) = \tan\left(2x - y^{\frac{1}{2}}\right) \]

\( g = \tan\left(2x - \sqrt{y}\right) \)  \hspace{1cm} (2.2.2.4)

A differential polynomial system (DPS) is given by:

\[ g_x = 2g^2 + 2, \quad g_y = \frac{g^4}{4y} + \frac{g^2}{2y} + \frac{1}{4y} \]  \hspace{1cm} (2.2.2.5)

\[ \text{pdetest}(2.2.4, (2.2.5)) \]

\[ [0, 0] \]  \hspace{1cm} (2.2.2.6)

\[ F\left(e^{\sqrt{z}}; k\right) \]  \hspace{1cm} (2.2.2.7)

\[ f(z, k) = F\left(e^{\sqrt{z}}; k\right), \quad \begin{aligned} f_{k, k, k} &= \left(3f_{k, z} + \frac{-5k^2 + 1}{k^3 - k}\right)f_{k, k} + \left(\frac{(9k^2 - 3)f_k}{(k^3 - k)f_z}\right) \\
+ &\quad -\frac{3f(z, k)}{(k^2 - 1)f_z}\right)f_{k, z} + \frac{-4k^2 - 1}{k^2(k^2 - 1)}, \quad f_{z, z} = \left(k^3(k - 1)^2(k + 1)^2 f_{k, k}^2 \right) \]  \hspace{1cm} (2.2.2.8)
Not just known functions but also arbitrary functions of algebraic expressions can be represented in differential polynomial

\[ F(x, y) = f \left( \exp \left( \sqrt{x} \right) \right) + g(y) \]  

\[ F(x, y) = f \left( \sqrt{x} \right) + g(y) \]  

\[ \left[ F_{x, y} = -\frac{F_x g_{y, y}}{g_y} + \frac{F_{x, y}}{F_y}, F_{x, x} = \left( \frac{2 F_x^2 F_{y, y}}{F_y^2} - \frac{2 F_x^2 g_{y, y}}{F_y g_y} - \frac{F_x^2}{F_y} \right) F_{x, x} \right] \]  

\[ + \left( \frac{F_{x, y}^2}{2 F_y^4} + \left( \frac{2 F_x^4 g_{y, y}}{F_y^4} + \frac{F_{x, y}^3}{F_y^2} x \right) F_{y, y} - \frac{F_x^4 g_{y, y}^2}{F_y^3 g_y} x - \frac{F_x^4 g_{y, y}^2}{F_y^3 g_y} x \right) F_{x, x} \]  

\[ + \left( \frac{x - 1}{4} F_{x, x}^2 \right) \]  

\[ \&where \left[ 2 F_{x, x} F_y g_y x - 2 F_{y, y} F_x^2 g_y + 2 g_{y, y} F_x^2 x F_y \right] \]  

\[ + F_x g_y F_x^2 \neq 0 \]

\[ \text{pdetest}('%', '%) \]

\[ [0, 0] \]
Branch cuts for algebraic expressions

> FunctionAdvisor(branch_cuts, arcsin(z), plot = 3 D)

What about the cuts of an algebraic expression?

> FunctionAdvisor(branch_cuts, arcsin(2 z \sqrt{1 - z^2}), plot = 3 D)
A classic example in the theory of branch cut calculation is that of Kahan's teardrop:

\[
\arcsin\left(2z \sqrt{-z^2 + 1}\right), \quad 1 < z, z < -1, \quad \Re(z) = -\frac{\sqrt{4\Re(z)^2 + 2}}{2}, \quad \Re(z)
\]

\[
= \frac{\sqrt{4\Im(z)^2 + 2}}{2}
\]

A classic example in the theory of branch cut calculation is that of Kahan's teardrop:

\[
KTD := 2 \text{arccosh}\left(\frac{(3 + (2 z))}{3}\right) - 2 \text{arccosh}\left(\frac{(12 + (5 z))}{3 (z + 4)}\right) - 2 \text{arccosh}\left(2 \left(z + 3\right) \sqrt{\frac{(z + 3)}{(27 (z + 4))}}\right)
\]

\[
\text{FunctionAdvisor}(\text{branch\_cuts}, \text{KTD}, \text{plot} = 3., \text{shift\_range} = -3, \text{scale\_range} = 2, \text{orientation} = [ -124, 20, 14])
\]
\[
\begin{align*}
\Re \left( 2 \arccosh \left( 1 + \frac{2}{3} z \right) 
- 2 \arccosh \left( \frac{12 + 5 z}{3 z + 12} \right) 
- 2 \arccosh \left( 2 z \right) 
+ 6 \sqrt{\frac{z + 3}{27 z + 108}} \right),
& \text{if } z \leq 0 \land -4 < z \land -\frac{9}{2} < \Im(z) < -4, \\
& \Re(z) \leq 0 \land -\frac{9}{4} < \Re(z), \\
& \Im(z) = -\sqrt{\frac{-4 \Re(z)^2 - 28 \Re(z) - 45 (\Re(z) + 3)}{2 \Re(z) + 5}} \land \Re(z) \leq -3 \land -\frac{9}{2} < \Re(z), \\
& \Im(z) = \sqrt{\frac{-4 \Re(z)^2 - 28 \Re(z) - 45 (\Re(z) + 3)}{2 \Re(z) + 5}} \land -\frac{9}{4} < \Re(z),
\end{align*}
\]
\[\begin{align*}
\Re(z) < -\frac{9}{4}, \Im(z) &= -\sqrt{-4 \Re(z)^2 - 28 \Re(z) - 45} \frac{(\Re(z) + 3)}{2 \Re(z) + 5} \land \Re(z) \\
\leq -3 \land -\frac{9}{2} < \Re(z), \Im(z) &= -\sqrt{-4 \Re(z)^2 - 28 \Re(z) - 45} \frac{(\Re(z) + 3)}{2 \Re(z) + 5} \\
-\frac{5}{2} < \Re(z) \land \Re(z) < -\frac{9}{4}
\end{align*}\]

**References**

### The \(n\)th derivative problem for algebraic expressions

A power where the exponent is linear in the differentiation variable, a relatively easy problem, can be in a database

```plaintext
> restart
> (%diff = diff) (f^α z + β, z^n)
```

More complicated, consider the \(k\)th power of a generic function; the corresponding symbolic derivative can be mapped into a sum of symbolic derivatives of powers of \(g(z)\) with lower degree

```plaintext
> (%diff = diff) (g(a z + β)^k, z^n) assuming k :: posint, k > n
```

For \(g = \ln\) this result can be expressed using Stirling numbers of the first kind

```plaintext
> (%diff = diff) (ln(a z + β)^k, z^n)
```

\[
\frac{\partial^n}{\partial z^n} \left( \ln(a z + \beta)^k \right) = \frac{\alpha^n \left( \sum_{k=0}^{n} (k - _{-}k_1 + 1)_{-}k_1 S_n^{k_1} \ln(a z + \beta)^{k - _{-}k_1} \right)}{(a z + \beta)^n}
\]
The case of the MeijerG function of an arbitrary number of parameters

\[
\text{MeijerG} \left( \left[ \left[ a[i] \right] \_i=1..n \right], \left[ b[i] \right] \_i=n+1..p \right], \left[ \left[ a[i] \right] \_i=1..m \right], \left[ b[i] \right] \_i=m+1..q \right) \left( z \right)
\]

\[
G^{m, n}_{p, q} \left( z \left| \begin{array}{c}
\begin{array}{c}
a_1, \ldots, a_n, b_{n+1}, \ldots, b_p \\
b_1, \ldots, b_m, b_{m+1}, \ldots, b_q
\end{array}
\end{array} \right. \right) = G^{m, n+1}_{1+p, q+1} \left( z \left| \begin{array}{c}
\begin{array}{c}
-k, a_1 - k, \ldots, a_n - k, b_{n+1} - k, \ldots, b_p - k \\
b_1 - k, \ldots, b_m - k, 0, b_{m+1} - k, \ldots, b_q - k
\end{array}
\end{array} \right. \right)
\]

(2.4.4)

\[
\frac{d^k}{dz^k} \text{MeijerG} \left( \left[ \left[ a[i] \right] \_i=1..n \right], \left[ b[i] \right] \_i=n+1..p \right], \left[ \left[ a[i] \right] \_i=1..m \right], \left[ b[i] \right] \_i=m+1..q \right) \left( z \right)
\]

(2.4.5)

Not only the mathematics of this result is correct: the object returned is actually computable (for given concrete values of \( n, p, m \) and \( q \))

Finally, the "holy grail" of this problem: the \( n^{th} \) derivative of a composite function \( f(g(z)) \) - an implementation of the **Faa di Bruno formula**. Building on top of the basic blocks of knowledge, Faa di Bruno is now a new, higher level, basic block:

\[
\frac{d^n}{dz^n} f(g(z)) = \sum_{k=0}^{n} \text{D}^{(k2)}(f) g(z) \text{IncompleteBellB} \left( n, k2, \frac{d}{dz} g(z) \right)
\]

(2.4.6)

Note the symbolic sequence of symbolic order derivatives of lower degree, both of \( f \) and \( g \), also within the arguments of the IncompleteBellB function. This is a very abstract formula ...

To see this at work, consider, for instance, a case where the \( n^{th} \) derivatives of \( f(z) \) and \( g(z) \) can both be computed, say \( f = \sin \), \( g = \cos \)

\[
\sin(\cos(\alpha z + \beta))
\]

(2.4.7)

\[
\frac{d^n}{dz^n} \sin(\cos(\alpha z + \beta)) = \sum_{k=0}^{n} \sin \left( \cos(\alpha z + \beta) + \frac{k2 \pi}{2} \right) \text{IncompleteBellB} \left( n, \_k2, \cos(\alpha z + \beta + \frac{\pi}{2}) \alpha, \ldots, \cos(\alpha z + \beta + \frac{\pi}{2} + \frac{m-1}{2}) \alpha^n - \_k2 + 1 \right)
\]

(2.4.8)

Take for instance \( n = 3 \)

\[
\frac{d^3}{dz^3} \sin(\cos(\alpha z + \beta)) = \sum_{k=0}^{3} \sin \left( \cos(\alpha z + \beta) + \frac{k2 \pi}{2} \right) \text{IncompleteBellB} \left( 3, \_k2, \cos(\alpha z + \beta + \frac{\pi}{2}) \alpha, \ldots, \cos(\alpha z + \beta + \frac{\pi}{2} + \frac{m-1}{2}) \alpha^3 - \_k2 \right)
\]

(2.4.9)
\[ \alpha^3 \sin(\alpha z + \beta) \cos(\cos(\alpha z + \beta)) - 3 \alpha^3 \cos(\alpha z + \beta) \sin(\alpha z) \\
+ \beta) \sin(\cos(\alpha z + \beta)) + \alpha^3 \sin(\alpha z + \beta)^3 \cos(\cos(\alpha z + \beta)) = \alpha^3 \sin(\alpha z) \\
+ \beta) \cos(\cos(\alpha z + \beta)) - 3 \alpha^3 \cos(\alpha z + \beta) \sin(\alpha z + \beta) \sin(\cos(\alpha z \\
+ \beta)) + \alpha^3 \sin(\alpha z + \beta)^3 \cos(\cos(\alpha z + \beta)) \\
The two sides of this equation are equal
\]

\[
> \text{simplify}(\text{lhs} - \text{rhs}) (2.4.10)) \\
0
\] (2.4.11)

### Conversion network for mathematical and algebraic expressions

#### Examples

```markdown
> restart;
Start with the error function
> erf(z)
\[
\text{erf}(z)
\] (2.5.1.1)

> convert((2.5.1.1), HermiteH)
\[
- \frac{2 H_{-1}(z)}{\sqrt{\pi} \: e^{z^2}} + 1
\] (2.5.1.2)

> convert((2.5.1.2), KummerU)
\[
z U\left(1, \frac{3}{2}, z^2\right) \sqrt{z^2} - \sqrt{\pi} \: e^{z^2} \left(z - \sqrt{z^2}\right) \\
- \frac{1}{\sqrt{z^2} \sqrt{\pi} \: e^{z^2}} + 1
\] (2.5.1.3)

> normal(convert((2.5.1.3), WhittakerW))
\[
z \left(-\sqrt{\pi} \: e^{z^2} \left(z^2\right)^{1/4} + W_{-1/4, 1/4} \left(z^2\right) e^{z^2}\right) \\
- \frac{1}{\sqrt{\pi} \: e^{z^2} \left(z^2\right)^{3/4}}
\] (2.5.1.4)

> convert((2.5.1.4), hypergeom)
\[
- \frac{1}{\sqrt{\pi} \: e^{z^2} \left(z^2\right)^{3/4}} \left(z \left(-\sqrt{\pi} \: e^{z^2} \left(z^2\right)^{1/4} - 2 \left(z^2\right)^{3/4} {}_1F_1\left(1; \frac{3}{2}; z^2\right) \right) + \left(z^2\right)^{1/4} \sqrt{\pi} \: _0F_0\left(\cdot; z^2\right) \right)
\] (2.5.1.5)

> simplify(2.5.1.5)
\[
\text{erf}(z)
\] (2.5.1.6)
```

Many "rule conversions" can be requested at once and performed in a specified order; for instance, consider the following expression:
\[ ee := \frac{1}{8} U\left(\frac{3}{2}, \frac{1}{2}, z^2\right) - \frac{1}{8} U\left(1, \frac{3}{2}, z^2\right) \sqrt{z^2} - \frac{1}{4} U\left(1, \frac{3}{2}, z^2\right) z^2 \sqrt{z^2} + \left(\frac{1}{4} \sqrt{z^2}\right) \]
\[ ee := \frac{U\left(\frac{3}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right)}{8} \sqrt{z^2} - \frac{U\left(1, \frac{3}{2}, z^2\right)}{4} z^2 \sqrt{z^2} \]  
(2.5.1.7)

By converting it to 'KummerU' and applying rules for normalizing the first and second indices, here respectively called \(a\) and \(b\), and then mixing them, you obtain varied representations for the same expression.

\[ convert\left(ee, \text{KummerU}, \"normalize a\", \"normalize b\", \"mix a and b\"\right) \]
\[ \frac{\sqrt{z^2}}{8} U\left(2, \frac{3}{2}, z^2\right) - \frac{U\left(1, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(1, \frac{1}{2}, z^2\right)}{4} z^2 \right) z^2 + \frac{\sqrt{z^2}}{4} \]  
(2.5.1.8)

\[ convert\left((2.5.1.8), \text{KummerU}, \"normalize a\", \"normalize b\", \"mix a and b\"\right) \]
\[ \frac{(2 z^2 + 1)}{8} U\left(1, \frac{1}{2}, z^2\right) - \frac{U\left(1, \frac{3}{2}, z^2\right)}{8} \sqrt{z^2} - \frac{U\left(1, \frac{3}{2}, z^2\right)}{4} z^2 \sqrt{z^2} \]  
(2.5.1.9)

A further manipulation actually shows the expression was always equal to zero :) \(2.5.1.10\)

\[ convert\left((2.5.1.9), \text{KummerU}, \"normalize a\", \"normalize b\", \"mix a and b\"\right) \]
\[ \frac{(2 z^2 + 1)}{8} U\left(1, \frac{1}{2}, z^2\right) - \frac{U\left(1, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(1, \frac{1}{2}, z^2\right)}{4} z^2 \]  
(2.5.1.10)

\[ normal\left((2.5.1.10)\right) \]
\[ 0 \]  
(2.5.1.11)

\[ General\ description \]

- The functions to which you can convert - the second argument could be one of these - are:

AiryAi AiBi arcacos arcacosh
arccot arccoth arcacsc arcacsch
arcsec arcsech arcasin arcasinh
arctan arctanh BellB BesselK
BesselU BesselK BesselY ChebyshevT
ChebyshevU cos cosh cot
coth CoulombF csc csch
CylinderD CylinderU CylinderV dilog
When a function class name is given as second argument the routines attempt to express expr using any of the functions of that class. The function classes understood by convert are:

- `'0F1'`
- `'1F1'`
- `'2F1'`
- `Airy`
- `arctrig`
- `arctigh`
- `Bessel_related`
- `Chebyshev`
- `Cylinder`
- `Ei_related`
- `elementary`
- `Elliptic_related`
- `erf_related`
- `GAMMA_related`
- `Hankel`
- `Heun`
- `Kelvin`
- `Legendre`
- `trig`
- `trigh`
- `Whittaker`
- `WhittakerW`
- `Wrightomega`

* (note spaces between words are filled with _ )

Despite the large number of function classes, most of the functions of mathematical physics belong to one of the three hypergeometric classes: `2F1`, `1F1`, and `0F1` where these three classes also include as particular cases all the elementary functions (trig, hyperbolic trig, their arcs, exp, and ln).

### The Optional Arguments

The conversion routines accept an optional extra argument indicating a rule; it could be one rule or a sequence of them:

- "raise a"
- "lower a"
- "normalize a"
- "raise b"
- "lower b"
- "normalize b"
- "raise c"
- "lower c"
- "normalize c"
- "mix a and b"
- "1F1 to 0F1"
- "0F1 to 1F1"
- "quadratic 1"
- "quadratic 2"
- "quadratic 3"
- "quadratic 4"
- "quadratic 5"
- "quadratic 6"
References