Magnetic traps in cold-atom physics

Pascal Szriftgiser[1] and Edgardo S. Cheb-Terrab[2]
(1) Laboratoire PhLAM, UMR CNRS 8523, Université Lille 1, F-59655, France
(2) Maplesoft

We consider a device constructed with a set of electrical wires fed with constant electrical currents. Those wires can have an arbitrary complex shape. The device is operated in a regime such that, in some region of interest, the moving particles experience a magnetic field that varies slowly compared to the Larmor spin precession frequency. In this region, the effective potential is proportional to the modulus of the field: $\|\vec{B}(x, y, z)\|$, this potential has a minimum and, close to this minimum, the device behaves as a magnetic trap.

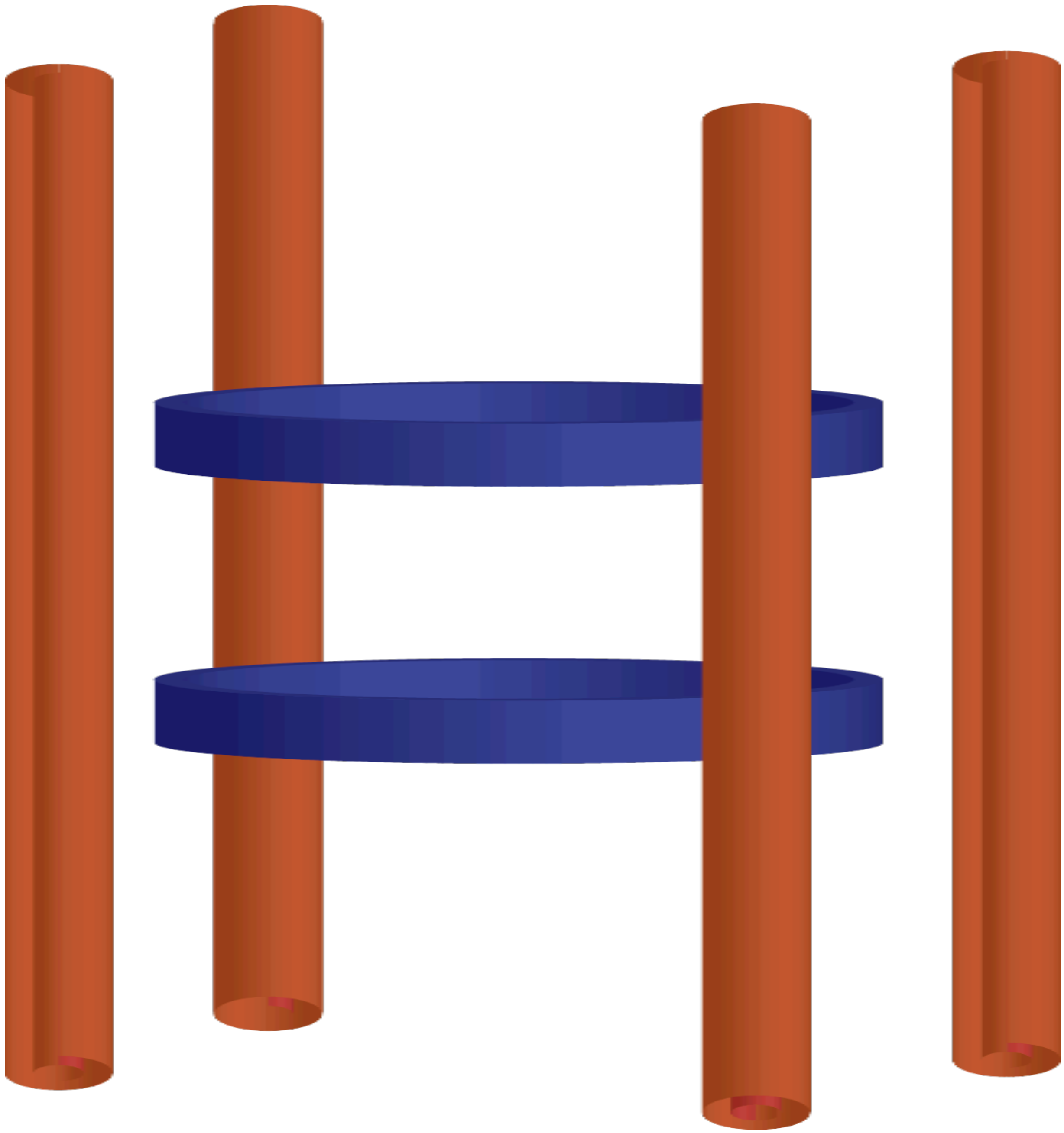


Figure 1: Schematic representation of a Ioffe-Pritchard magnetic trap. It is made of four infinite rods and two coils.

Following [1], we show that:

a) For a time-independent magnetic field $\vec{B}(x, y, z)$ in vacuum, up to order two in the relative coordinates $X_i = [x, y, z]$ around some point of interest, the coefficients of orders 1 and 2 in this expansion, $v_{i,j}$ and $c_{i,j,k}$, respectively the gradient and curvature, contain only 5 and 7 independent components.

b) All stationary points of $\| \vec{B}(x, y, z) \|^2$ (nonzero minima and saddle points) are confined to a curved surface defined by $\det(\partial_j(B_i)) = 0$.

c) The effective potential, proportional to $\| \vec{B}(x, y, z) \|$, has no maximum, only a minimum.

Finally, we draw the stationary condition surface for the case of the widely used Ioffe-Pritchard magnetic trap.

Reference

[1] R. Gerritsma and R. J. C. Spreeuw, ***Topological constraints on magnetostatic traps***, Phys. Rev. A **74**, 043405 (2006)

The independent components of $v_{i,j}$ and $c_{i,j,k}$ entering $B_i = u_i + v_{i,j} X_j + \frac{1}{2} c_{i,j,k} X_j X_k$

```
> restart
> with(Physics) :
> Setup(coordinates = cartesian, dimension = 3, metric = Euclidean, spacetimeindices
      = lowercaselatin, quiet, minimizetensorcomponents = true)
```

$$[coordinatesystems = \{X\}, dimension = 3, metric = \{(1, 1) = 1, (2, 2) = 1, (3, 3) = 1\}, minimizetensorcomponents = true, spacetimeindices = lowercaselatin] \quad (1)$$

```
> g_[ ]
```

$$g_{a, b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

We are interested in determining the location of the stationary points of $\| \vec{B}(x, y, z) \|^2$, around which the device behaves as a magnetic trap.

Up to order two in the relative coordinates $X_i = (x, y, z)$ around a point of interest x_i within this region (that we take as origin of the system of references),

```
> 𝔹[i] = u[i] + v[i, j] X[j] + 1/2 c[i, j, k] X[j] X[k]
```

$$\mathbb{B}_i = u_i + v_{i,j} X_j + \frac{1}{2} c_{i,j,k} X_j X_k \quad (3)$$

where $\mathbb{B}_i$ is the truncated expansion of B_i (computers are picky, don't like recursive definitions) and, following [1], we introduce the notation

$$u_i = B_i(X)\Big|_{X_j = 0}$$

$$v_{i,j} = \partial_j\big(B_i(X)\big)\Big|_{X_j = 0}$$

$$c_{i,j,k} = \partial_j\big(\partial_k\big(B_i(X)\big)\big)\Big|_{X_j = 0}$$

Here, $v_{i,j}$ denotes the gradient tensor and $c_{i,j,k}$ the curvature of $\vec{B}$ at the point $x_i = 0$.

> *CompactDisplay*(𝔹[*i*](*X*), *B*[*i*](*X*))

$\mathbb{B}(X)$ *will now be displayed as* $\mathbb{B}$

$B(X)$ *will now be displayed as* B **(4)**

> *Define*(*B*[*i*], **(3)**)

Defined objects with tensor properties

$$\left\{\mathbb{B}_i, B_i, \gamma_a, \sigma_a, X_a, c_{i,j,k}, \partial_a, g_{a,b}, u_i, v_{i,j}, \delta_{a,b}, \epsilon_{a,b,c}\right\} \quad \textbf{(5)}$$

At this stage, the gradient tensor $v_{i,j}$ has no known symmetry; it thus has up to 9 independent components and, in general, a rank 3 tensor like $c_{i,j,k}$ has $3^3 = 27$ independent components,

> *Library*:-*NumberOfIndependentTensorComponents*(*v*);

9 **(6)**

> *Library*:-*NumberOfIndependentTensorComponents*(*c*)

27 **(7)**

but $c_{i,j,k} = \partial_j\big(\partial_k\big(B_i(X)\big)\big)\Big|_{X_j = 0}$ is invariant by a permutation of its second and third indices; indicate this symmetry and this last number is reduced to 18:

> *Define*(*redo*, *c*[*i*, *j*, *k*], *symmetric* = {2, 3}, *quiet*)

$$\left\{\mathbb{B}_i, B_i, \gamma_a, \sigma_a, X_a, c_{a,b,d}, \partial_a, g_{a,b}, u_i, v_{i,j}, \delta_{a,b}, \epsilon_{a,b,c}\right\} \quad \textbf{(8)}$$

> *Library*:-*NumberOfIndependentTensorComponents*(*c*)

18 **(9)**

> *c*[1, *j*, *k*, *matrix*]

$$c_{1,j,k} = \begin{bmatrix} c_{1,1,1} & c_{1,1,2} & c_{1,1,3} \\ c_{1,1,2} & c_{1,2,2} & c_{1,2,3} \\ c_{1,1,3} & c_{1,2,3} & c_{1,3,3} \end{bmatrix} \quad \textbf{(10)}$$

For a stationary magnetic field $\vec{B}$ in vacuum, we also have

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = 0$$

so that the number of independent components of $c_{i,j,k}$ can be further reduced.

>

$\nabla \times \vec{B} = 0$

Starting with $\nabla \times \vec{B} = 0$, from **(3)**

> **(3)**

$$\mathbb{B}_i = u_i + v_{i,j} X_j + \frac{1}{2} c_{i,j,k} X_j X_k \quad \textbf{(11)}$$

> *LeviCivita*[a, b, i]·d_[b](**(3)**)

$$\epsilon_{a,b,i} \partial_b(\mathbb{B}_i) = \epsilon_{a,b,i} \left(v_{i,j} g_{b,j} + \frac{c_{i,j,k} (X_j g_{b,k} + g_{b,j} X_k)}{2} \right) \quad \textbf{(12)}$$

> *Simplify*(**(12)**)

$$\epsilon_{a,b,i} \partial_b(\mathbb{B}_i) = (-X_j c_{i,j,k} - v_{i,k}) \epsilon_{a,i,k} \quad \textbf{(13)}$$

The right-hand-side of **(13)** must be zero no matter what the X_j are. This means that the following two terms, the coefficients of X_j, are equal to 0.

> *Coefficients*(**(13)**, X[j])

$$\epsilon_{a,b,i} \partial_b(\mathbb{B}_i) = -\epsilon_{a,i,k} v_{i,k}, 0 = -c_{i,j,k} \epsilon_{a,i,k} \quad \textbf{(14)}$$

From the first of these equations, one can then see that $v_{i,j}$ is actually symmetric:

> 0 = *rhs*(**(14)**[1])

$$0 = -\epsilon_{a,i,k} v_{i,k} \quad \textbf{(15)}$$

> *TensorArray*(%)

$$\begin{bmatrix} 0 = -v_{2,3} + v_{3,2} & 0 = v_{1,3} - v_{3,1} & 0 = -v_{1,2} + v_{2,1} \end{bmatrix} \quad \textbf{(16)}$$

Add this symmetry to the definition of $v_{i,j}$

> *Define*(*redo*, v[i, j], *symmetric*, *quiet*)

$$\{\mathbb{B}_i, B_i, \gamma_a, \sigma_a, X_a, c_{a,b,d}, \partial_a, g_{a,b}, u_i, v_{a,b}, \delta_{a,b}, \epsilon_{a,b,c}\} \quad \textbf{(17)}$$

Check that the symmetry is explicitly there

> v[ ]

$$v_{a,b} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{1,2} & v_{2,2} & v_{2,3} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{bmatrix} \quad \textbf{(18)}$$

> *Library:-NumberOfIndependentTensorComponents*(v)

$$6 \quad \textbf{(19)}$$

From the second equation in **(14)**,

> **(14)**[2]

$$0 = -c_{i,j,k}\,\epsilon_{a,i,k} \qquad \textbf{(20)}$$

$c_{i,j,k}$ is also symmetric under permutation of its 1st and 3rd indices, and because it is already symmetric under permutation of its 2nd and 3rd indices, $c_{i,j,k}$ is actually fully symmetric. Although this is sort of obvious, it can be verified as follows: redefine $c_{i,j,k}$ indicating the symmetries {1, 3}, {2, 3} and check the symmetries of the resulting tensor:

> *Define*(*redo*, *c*[*i*, *j*, *k*], *symmetric* = {{1, 3}, {2, 3}})

Defined objects with tensor properties

$$\left\{\mathbb{B}_i, B_i, \gamma_a, \sigma_a, X_a, c_{a,b,d}, \partial_a, g_{a,b}, u_i, v_{a,b}, \delta_{a,b}, \epsilon_{a,b,c}\right\} \qquad \textbf{(21)}$$

> *Library*:-*GetTensorSymmetryProperties*(*c*)

$$\{[1, 2, 3]\}, \varnothing \qquad \textbf{(22)}$$

Or, directly count the number of independent components:

> *Library*:-*NumberOfIndependentTensorComponents*(*c*)

$$10 \qquad \textbf{(23)}$$

Indeed, a fully symmetric tensor constitutes a vector space with a dimension given by the binomial of the dimension + rank - 1 and the rank; that is

> $VectorSpaceDim := (dimension, rank) \rightarrow \binom{dimension + rank - 1}{rank}$:

> *VectorSpaceDim*(3, 3)

$$10 \qquad \textbf{(24)}$$

>

$\nabla \cdot \vec{B} = 0$

Again, starting from **(3)**

> **(3)**

$$\mathbb{B}_i = u_i + v_{i,j} X_j + \frac{1}{2} c_{i,j,k} X_j X_k \qquad \textbf{(25)}$$

> *d_*[*i*](**(3)**)

$$\partial_i(\mathbb{B}_i) = v_{i,j} g_{i,j} + \frac{c_{i,j,k}\left(X_j g_{i,k} + g_{i,j} X_k\right)}{2} \qquad \textbf{(26)}$$

> *Simplify*(**(26)**)

$$\partial_i(\mathbb{B}_i) = X_k c_{j,j,k} + v_{j,j} \qquad \textbf{(27)}$$

The right-hand-side of **(27)** must be zero no matter what the X_j are. This means that the following two terms, the coefficients of X_k, are equal to 0.

> *Coefficients*(**(27)**, *X*[*k*])

$$\partial_i(\mathbb{B}_i) = v_{j,j},\ 0 = c_{j,j,k} \qquad \textbf{(28)}$$

This time the resulting equations don't increase the symmetry of the tensors as in $\nabla \times \vec{B} = 0$, but permit reducing the number of independent components.

From the first of these equations, $v_{i,j}$ is traceless

> $v[trace] = 0$

$$v_{1,1} + v_{2,2} + v_{3,3} = 0 \tag{29}$$

> $isolate(\mathbf{(29)}, v[3, 3])$

$$v_{3,3} = -v_{1,1} - v_{2,2} \tag{30}$$

> $Library{:}\text{-}RedefineTensorComponent(\mathbf{(30)})$

$$v_{a,b} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{1,2} & v_{2,2} & v_{2,3} \\ v_{1,3} & v_{2,3} & -v_{1,1} - v_{2,2} \end{bmatrix} \tag{31}$$

> $Library{:}\text{-}NumberOfIndependentTensorComponents(v)$

$$5 \tag{32}$$

From the second equation in **(28)**,

> $TensorArray(\mathbf{(28)}[2])$

$$[0 = c_{1,1,1} + c_{1,2,2} + c_{1,3,3}, 0 = c_{1,1,2} + c_{2,2,2} + c_{2,3,3}, 0 = c_{1,1,3} + c_{2,2,3} + c_{3,3,3}] \tag{33}$$

Redefine one component using each of these equations:

> $map(u \rightarrow isolate(u, op(-1, rhs(u))), \mathbf{(33)})$

$$\left[c_{1,3,3} = -c_{1,1,1} - c_{1,2,2} \quad c_{2,3,3} = -c_{1,1,2} - c_{2,2,2} \quad c_{3,3,3} = -c_{1,1,3} - c_{2,2,3} \right] \tag{34}$$

> $Library{:}\text{-}RedefineTensorComponent(\mathbf{(34)})$

$$\left[c_{a,b,d} = \left[\begin{array}{c} \textit{1..3 x 1..3 x 1..3 Array} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array} \right] \right] \tag{35}$$

> $c[i, j, 3, matrix]$

$$c_{i,j,3} = \begin{bmatrix} c_{1,1,3} & c_{1,2,3} & -c_{1,1,1} - c_{1,2,2} \\ c_{1,2,3} & c_{2,2,3} & -c_{1,1,2} - c_{2,2,2} \\ -c_{1,1,1} - c_{1,2,2} & -c_{1,1,2} - c_{2,2,2} & -c_{1,1,3} - c_{2,2,3} \end{bmatrix} \tag{36}$$

Now count the number of independent components of the curvature:

> $Library{:}\text{-}NumberOfIndependentTensorComponents(c)$

$$7 \tag{37}$$

By the way, due to the symmetries of the curvature, $\vec{B}$ is not just Curl and Divergence free, but also Laplacian free. From **(3)**,

> $SubstituteTensor\left(\mathbb{B}_i = B[i](X), \mathbf{(3)}\right)$

$$B_i = u_i + v_{i,j} X_j + \frac{1}{2} c_{i,j,k} X_j X_k \quad \textbf{(38)}$$

> *dAlembertian*(**(38)**)

$$\Box(B_i) = c_{i,k,k} \quad \textbf{(39)}$$

> *TensorArray*(**(39)**)

$$\left[\ \Box(B_1) = 0 \quad \Box(B_2) = 0 \quad \Box(B_3) = 0\ \right] \quad \textbf{(40)}$$

>

The stationary points are within the surface $\det\left(\partial_j\left(B_i\right)\right) = 0$

In order to determine the location of the stationary points of the square of the potential, $U \sim \|B\|^2$, we need a copy of $\mathbb{B}_i$ with different repeated dummy indices.

> *SubstituteTensorIndices*({$j = m, k = n$}, **(3)**)

$$\mathbb{B}_i = u_i + v_{i,m} X_m + \frac{1}{2} c_{i,m,n} X_m X_n \quad \textbf{(41)}$$

> **(3)** · **(41)**

$$\mathbb{B}_i^2 = \left(u_i + v_{i,j} X_j + \frac{1}{2} c_{i,j,k} X_j X_k\right)\left(u_i + v_{i,m} X_m + \frac{1}{2} c_{i,m,n} X_m X_n\right) \quad \textbf{(42)}$$

> *Simplify*(**(42)**)

$$\mathbb{B}_i^2 = \left(\frac{1}{4} X_k X_a X_m X_n c_{i,m,n} + X_k X_a u_i + X_j X_k X_a v_{i,j}\right) c_{a,i,k} + v_{i,j} X_j v_{i,m} X_m + 2 v_{i,j} X_j u_i + u_i^2 \quad \textbf{(43)}$$

Removing higher order terms (> 2) with respect to the coordinates,

> *STV* := [*op*(*indets*(**(43)**, *specfunc*(*SpaceTimeVector*)))]

$$STV := \left[X_a, X_j, X_k, X_m, X_n\right] \quad \textbf{(44)}$$

> *select*($u \rightarrow$ *degree*(u, *STV*) :: *identical*(3, 4), *expand*(*rhs*(**(43)**)))

$$\frac{1}{4} X_k X_a X_m X_n c_{a,i,k} c_{i,m,n} + X_j X_k X_a c_{a,i,k} v_{i,j} \quad \textbf{(45)}$$

> $U(X)$ = *Simplify*(*rhs*(**(43)**) − **(45)**) :
> *SubstituteTensorIndices*({$a = i, m = k$}, *expand*(%))

$$U(X) = X_j X_k c_{i,j,k} u_i + v_{i,j} v_{i,k} X_j X_k + 2 v_{i,j} X_j u_i + u_i^2 \quad \textbf{(46)}$$

> *CompactDisplay*($U(X)$)

U(X) will now be displayed as U **(47)**

> *collect*(**(46)**, *STV*, *distributed*)

$$U = \left(c_{i,j,k} u_i + v_{i,j} v_{i,k}\right) X_j X_k + 2 v_{i,j} X_j u_i + u_i^2 \quad \textbf{(48)}$$

For U to be stationary, all of its first derivatives $\partial_p(U)$ must cancel at $X = 0$.

> *d_*[*n*](**(48)**)

$$\partial_n(U) = \left(c_{i,j,k} u_i + v_{i,j} v_{i,k}\right)\left(X_j g_{k,n} + g_{j,n} X_k\right) + 2 v_{i,j} u_i g_{j,n} \quad \textbf{(49)}$$

> *Simplify*((**49**))

$$\partial_n(U) = \left(2 X_j c_{i,j,n} + 2 v_{i,n}\right) u_i + 2 X_j v_{i,j} v_{i,n} \quad \textbf{(50)}$$

Evaluation at $X = 0$:

> *SubstituteTensor*($X[j] = 0$, (**50**))

$$\partial_n(U) = 2 v_{i,n} u_i \quad \textbf{(51)}$$

Except for the trivial solution $u_i = 0$, for U to be stationary, u_i must be an eigenvector of $v_{i,j}$ with eigenvalue 0.

Now, the determinant of $v_{i,j}$ is the product of its eigenvalues, hence the stationary points occur where $det\left(v_{i,j}\right) = 0$. In turn, $v_{i,j} = \partial_j\left(B_i\left(x_k\right)\right)$ where x_k is some point within the magnetic trap, hence the stationary points are the x_k of the 2D surface

$$\det\left(\partial_j\left(B_i\right)\right) = 0$$

>

$U = \left\|\vec{B}\right\|^2$ has only minima, no maxima

To see that U has no maxima, only minima, we need to insert $u_i v_{i,n} = 0$ in the definition **(46)** of U and consider the second derivative with respect to the coordinates:

> (**46**)

$$U = X_j X_k c_{i,j,k} u_i + v_{i,j} v_{i,k} X_j X_k + 2 v_{i,j} X_j u_i + u_i^2 \quad \textbf{(52)}$$

> $0 = v[i,j]\,u[i]$

$$0 = u_i v_{i,j} \quad \textbf{(53)}$$

> 2 (**53**) $X[j]$

$$0 = 2 v_{i,j} X_j u_i \quad \textbf{(54)}$$

> (**46**) − (**54**)

$$U = X_j X_k c_{i,j,k} u_i + v_{i,j} v_{i,k} X_j X_k + u_i^2 \quad \textbf{(55)}$$

The second derivative is given by the coefficient with respect to $X_k X_j$

> $t[j,k] =$ *Coefficients*(*rhs*((**55**)), $X[j]\cdot X[k]$, 1)

$$t_{j,k} = c_{i,j,k} u_i + v_{i,j} v_{i,k} \quad \textbf{(56)}$$

Now, U has only minima, no maxima, if this second derivative is always positive (its trace is positive definite). So take the trace of this expression:

> *Define*((**56**))

Defined objects with tensor properties

$$\left\{\mathbb{B}_i, B_i, \gamma_a, \sigma_a, X_a, c_{a,b,d}, \partial_a, g_{a,b}, t_{j,k}, u_i, v_{a,b}, \delta_{a,b}, \epsilon_{a,b,c}\right\} \quad \textbf{(57)}$$

> $t[trace]$

$$2 v_{1,1}^2 + 2 v_{1,1} v_{2,2} + 2 v_{1,2}^2 + 2 v_{1,3}^2 + 2 v_{2,2}^2 + 2 v_{2,3}^2 \quad \textbf{(58)}$$

Add and subtract:

> $(v[1, 1] + v[2, 2])^2$

$$\left(v_{1,1} + v_{2,2}\right)^2 \tag{59}$$

> $t[trace] + \mathbf{(59)} - expand(\mathbf{(59)})$

$$v_{1,1}^2 + 2\, v_{1,2}^2 + 2\, v_{1,3}^2 + v_{2,2}^2 + 2\, v_{2,3}^2 + \left(v_{1,1} + v_{2,2}\right)^2 \tag{60}$$

This trace is a sum of the squares of real quantities. It is therefore always positive. This recovers a well known result: there is no local maximum of a static magnetic field in free space. Indeed, such a maximum would require the three eigenvalues of $v_{i,j}$ to be negative, which would also imply a negative trace. Therefore, the stationary condition can only be a local minimum or a saddle point. See [1] for a more thorough discussion.

>

Drawing the Ioffe-Pritchard Magnetic Trap

The magnetic field of the Ioffe-Pritchard trap, quadratic in the relative coordinates [x,y,z], is approximated as

> $B[j] = \begin{bmatrix} 0 \\ 0 \\ U \end{bmatrix} + A \cdot \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix} + \frac{C}{2} \cdot \begin{bmatrix} -x \cdot z \\ -y \cdot z \\ z^2 - \frac{1}{2} \cdot \left(x^2 + y^2\right) \end{bmatrix}$

$$B_j = \begin{bmatrix} A\,x - \frac{1}{2}\, C\,x\,z \\ -A\,y - \frac{1}{2}\, C\,y\,z \\ U + \frac{C\left(z^2 - \frac{x^2}{2} - \frac{y^2}{2}\right)}{2} \end{bmatrix} \tag{61}$$

> $Define(\mathbf{(61)})$

Defined objects with tensor properties

$$\left\{\mathbb{B}_i, B_i, \gamma_a, \sigma_a, X_a, c_{a,b,d}, \partial_a, g_{a,b}, t_{j,k}, u_i, v_{a,b}, \delta_{a,b}, \epsilon_{a,b,c}\right\} \tag{62}$$

The surface of stationary points is defined by

> $d\_[j](B[i])$

$$\partial_j\left(B_i\right) \tag{63}$$

The matrix behind:

> $TensorArray(\mathbf{(63)})$

(64)

$$\begin{bmatrix} A - \frac{C z}{2} & 0 & -\frac{C x}{2} \\ 0 & -A - \frac{C z}{2} & -\frac{C y}{2} \\ -\frac{C x}{2} & -\frac{C y}{2} & C z \end{bmatrix} \quad \textbf{(64)}$$

The stationary condition $\det\left(\partial_j\left(B_i\right)\right) = 0$:

> *factor*(*LinearAlgebra*:-*Determinant*(**(64)**)) = 0

$$-\frac{C\left(-C^2 x^2 z - C^2 z y^2 - 2 C^2 z^3 - 2 C x^2 A + 2 A C y^2 + 8 z A^2\right)}{8} = 0 \quad \textbf{(65)}$$

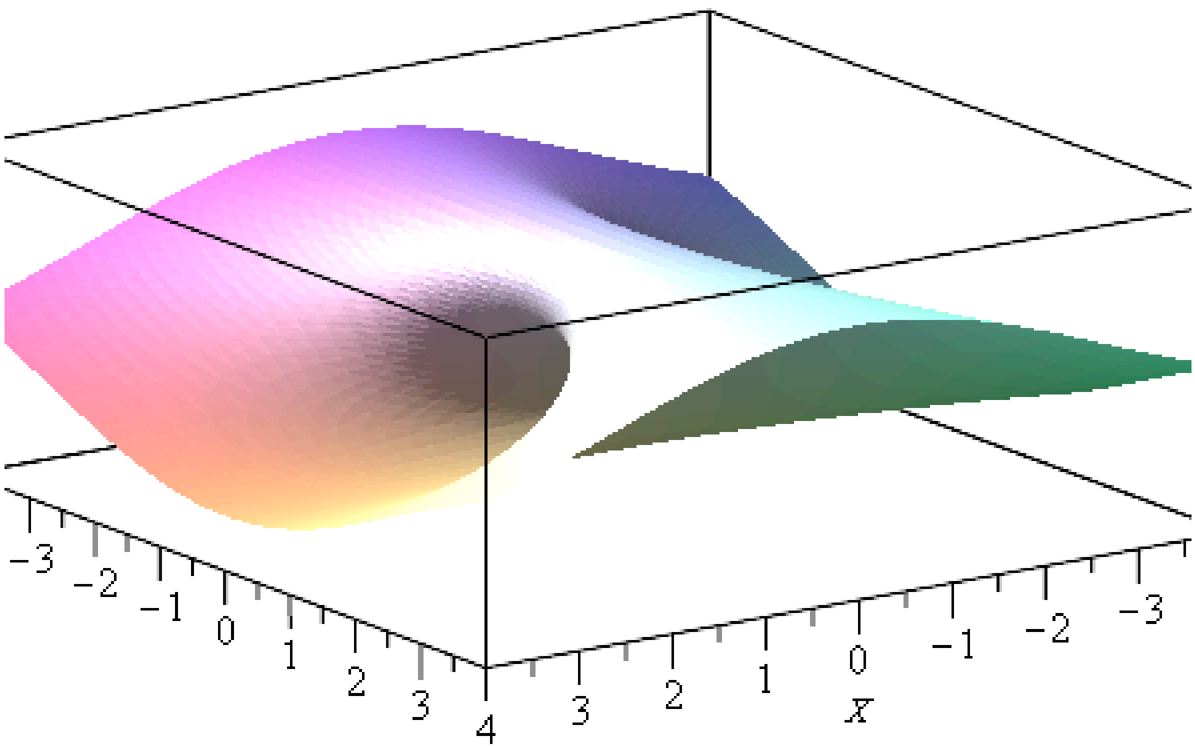
is scaled by a single parameter $\epsilon = \frac{2\ A}{C}$.

> *simplify*$\left(\frac{8}{C^3}\right.$ **(65)**, $\left.\left\{\frac{2\ A}{C} = \epsilon\right\}\right)$

$$2 z^3 + \left(-2 \epsilon^2 + x^2 + y^2\right) z + \epsilon \left(x^2 - y^2\right) = 0 \quad \textbf{(66)}$$

from which we can deduce the 2D stationary manifold. For $\epsilon = 1$:

```
> plots:-implicitplot3d(
  subs(epsilon = 1, (66)),
  x = -4 ..4, y = -4 ..4, z = -1.5 ..1.5,
  style = surface, scaling = constrained, grid = [50, 50, 50],
  caption = (The Ioffe-Pritchard magnetic trap for  epsilon = 1));
```



The Ioffe-Pritchard magnetic trap for $\epsilon = 1$

>