THE DYNAMICS OF AN ELLIPSE ROLLING WITHOUT SLIPPING ON A HORIZONTAL LINE WITHIN A VERTICAL PLANE

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ABSTRACT. We derive the differential equation of motion of a homogeneous ellipse which rolls without slipping on a horizontal line within a vertical plane.

1. Introduction

In this note we derive the equations of motion of a homogeneous elliptic lamina of mass m and the major and minor axes of lengths of a and b which rolls without slipping along the horizontal x axis within the vertical xy plane.

In sections 2 and 3 we investigate some geometric properties of an ellipse that are relevant to this study. In section 4 we apply the Lagrangian formalism to derive the differential equations of motion.

The elementary geometric analysis presented here contains the germ of the idea for extending the study to the three-dimensional case of a smooth solid object rolling without slipping on a horizontal plane but the details are significantly more complex there. In particular, the Lagrangian approach is not the best choice in that case since the no-slip rolling constraint is *nonholonomic* in three dimensions. The preferred method is Appell's formulation of dynamics which is quite more effective with nonholonomic constraints. Some day I may write a followup article on that.

2. The analysis of the geometry – Part I

Figure 1 depicts the rolling ellipse at a generic instance. The ellipse's orientation is measured through the angle α between its major axis and the x axis. The ellipse's point of contact with the x axis is marked P, while the foot of the perpendicular from the center C onto the x axis is marked Q. The goal of this section is to prove:

Proposition 1. *The distance $\sigma(\alpha)$ of the ellipse's center above the x axis is given by*

$$\sigma(\alpha) = \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 a}. \tag{1}$$

Proof. For the purpose of this proof it suffices to restrict the range of the angle α to the interval 0 to π since, as is evident in Figure 1, within that range the ellipse achieves all of its possible orientations. Calculating in terms of the rotated ellipse of Figure 1, however, is rather inconvenient. To simplify, we rotate the entire diagram so that the ellipse's major axis becomes horizontal, and introduce a new Cartesian

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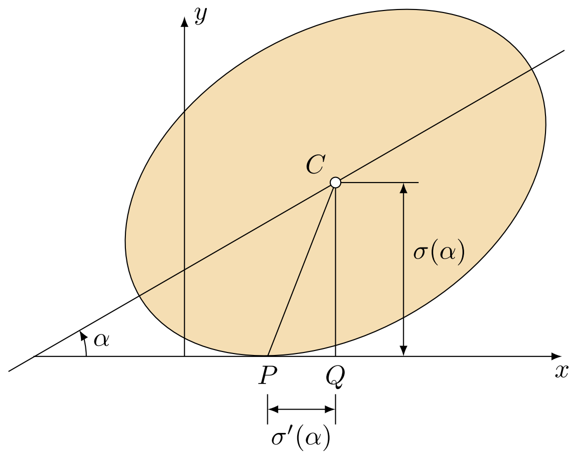

FIGURE 1. The ellipse rolls without slipping on the x axis with the vertical xy plane.

coordinate system $\xi\eta$ with the origin at the ellipse's center as shown in Figure 2. Then the equation of the ellipse takes the standard form

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1,$$

and what used to be the x axis now appears as the tangent line with the equation[1]

$$\xi \sin\alpha + \eta\cos\alpha = -\sigma(\alpha), \tag{2}$$

where $\sigma(\alpha) > 0$ is to be determined.

Let $(\bar\xi, \bar\eta)$ be the coordinates of the point of tangency P. The vector $\langle \bar\xi/a^2, \bar\eta/b^2\rangle$ is normal to the ellipse at P, and therefore it is parallel to the vector $\langle \sin\alpha, \cos\alpha\rangle$ which is normal to the tangent line. It follows that

$$\frac{\bar\xi}{a^2}\cos\alpha = \frac{\bar\eta}{b^2}\sin\alpha. \tag{3a}$$

At the same time, $(\bar\xi, \bar\eta)$ satisfies the equation of the ellipse

$$\frac{\bar\xi^2}{a^2} + \frac{\bar\eta^2}{b^2} = 1. \tag{3b}$$

From (3a) we have

$$\bar\eta = \frac{b^2}{a^2}\frac{\cos\alpha}{\sin\alpha}\bar\xi.$$

Substituting this in (3b) we obtain

$$\frac{\bar\xi}{a^2} + \frac{b^2}{a^4}\frac{\cos^2\alpha}{\sin^2\alpha}\bar\xi^2 = 1,$$

whence

$$\bar\xi^2 = \frac{a^4\sin^2\alpha}{a^2\sin^2\alpha + b^2\cos^2\alpha},$$

[1] In general, the distance of either of the lines $ax + by = \pm c$ from the origin in the Cartesian xy plane is $d = |c|/\sqrt{a^2+b^2}$. If $a^2 + b^2 = 1$, as it is in our case, then $|c|$ is the distance.

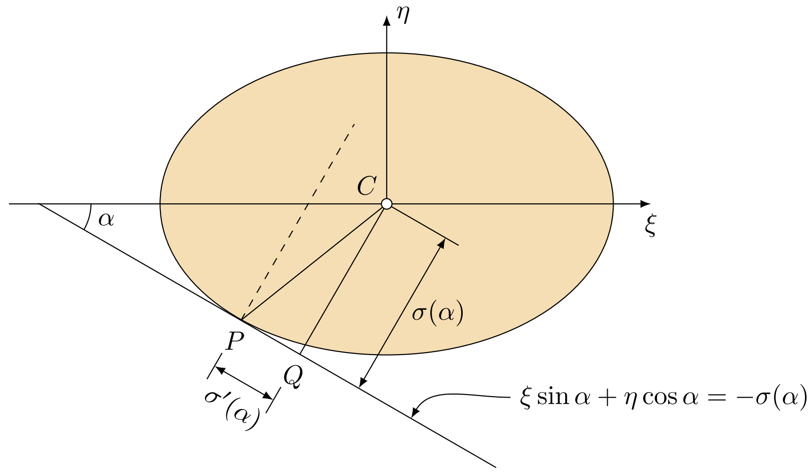

FIGURE 2. We rotate and translate the diagram in Figure 1 so that the ellipse takes on the standard configuration with whose equation is $\xi^2/a^2 + \eta^2/b^2 = 1$.

and since within the range $0 \le \alpha \le \pi$ we have $\bar{\xi} \le 0$ (see Figure 2) we conclude that

$$\bar{\xi} = -\frac{a^2 \sin\alpha}{\sqrt{a^2 \sin^2\alpha + b^2\cos^2\alpha}}.$$

and then from (3a)

$$\bar{\eta} = -\frac{b^2 \cos\alpha}{\sqrt{a^2 \sin^2\alpha + b^2\cos^2\alpha}}.$$

Substituting these in the equation of tangent line (2) yields (1). □

3. THE ANALYSIS OF THE GEOMETRY – PART II

In the previous section we calculated the height $\sigma(\alpha)$ of the ellipse's center above the x axis. Refer to Figure 1. In this section we calculate the distance between the point of tangency, P, and the foot of the perpendicular, Q. We show that

Proposition 2. *The (signed) length of the line segment PQ in Figure 1 is $\sigma'(\alpha)$, where $\sigma(\alpha)$ is given in* (1)*, and the prime indicates the derivative.*

Proof. As in the previous section, we prefer to work with the rotated and translated diagram of Figure 2. Whereas previously we wrote $(\bar{\xi}, \bar{\eta})$ for the coordinates of the contact point P, in this section we use the notation $\big(\xi(\alpha), \eta(\alpha)\big)$ since we will be investigating the dependence of the point of tangency on the angle α.

The point $\big(\xi(\alpha), \eta(\alpha)\big)$ lies both on the ellipse and the tangent line. Therefore

$$\frac{\xi(\alpha)^2}{a^2} + \frac{\eta(\alpha)^2}{b^2} = 1, \tag{4}$$

$$\xi(\alpha)\sin\alpha + \eta(\alpha)\cos\alpha = -\sigma(\alpha), \tag{5}$$

for all α.

Upon differentiating these two equations with respect to α we arrive at a linear system of two equations in the two unknowns $\xi'(\alpha)$ and $\eta'(\alpha)$:

$$\begin{bmatrix} \frac{2\xi(\alpha)}{a^2} & \frac{2\eta(\alpha)}{b^2} \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \xi'(\alpha) \\ \eta'(\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ -\xi(\alpha)\cos\alpha + \eta(\alpha)\sin\alpha - \sigma'(\alpha) \end{bmatrix}.$$

The rows of the coefficient matrix are the normal vectors to the ellipse and the tangent line. But at the point of tangency those normal vectors are parallel, therefore the coefficient matrix is singular. The only way that the system can be consistent is to have the right-hand side of the equation be zero, that is

$$\xi(\alpha)\cos\alpha - \eta(\alpha)\sin\alpha = -\sigma'(\alpha). \tag{6}$$

Solving this together with (5) we obtain the coordinates $\big(\xi(\alpha), \eta(\alpha)\big)$ of the point of tangency in terms of $\sigma(\alpha)$ and $\sigma'(\alpha)$:

$$\begin{aligned} \xi(\alpha) &= -\sigma(\alpha)\sin\alpha - \sigma'(\alpha)\cos\alpha, \\ \eta(\alpha) &= -\sigma(\alpha)\cos\alpha + \sigma'(\alpha)\sin\alpha. \end{aligned}$$

The geometrical interpretation of this calculation is that $\big(\xi(\alpha), \eta(\alpha)\big)$ is the intersection point of the lines (5) and (6). In Figure 2 the latter is shown as a dashed line. We note, however, that the distance of that line from the origin is $\sigma'(\alpha)$ (see footnote 1) but that distance is the length of PQ, and that concludes the proof. $\square$

4. The rolling dynamics of the ellipse

Referring to Figure 1, let (x_c, y_c) be the coordinates of the ellipse's center, C. Since $y_c = \sigma(\alpha)$, for the coordinates of the points C, P and Q we get:

$$C : \big(x_c, \sigma(\alpha)\big) \qquad P : \big(x_c - \sigma'(\alpha), 0\big), \qquad Q : \big(x_c, 0\big).$$

In particular, the vector CP is $\big\langle -\sigma'(\alpha), -\sigma(\alpha)\big\rangle$.

In order to apply the no-slip constraint, it is helpful to temporarily embed the two-dimensional xy plane into the three-dimensional space by adding a third coordinate z which is perpendicular to the xy plane and is so that xyz makes a right-handed coordinate system. Within that extended coordinate system the vector CP is $\big\langle -\sigma'(\alpha), -\sigma(\alpha), 0\big\rangle$ while the ellipse's angular velocity is $\omega = \langle 0, 0, \dot\alpha\rangle$.[2] Then the velocity, relative to the ellipse's center, of the material point on the ellipse which instantaneously coincides with the point of tangency, P, is given by the cross product $\omega \times CP = \big\langle \sigma(\alpha), -\sigma'(\alpha), 0\big\rangle\dot\alpha$, which as expected lies within the xy plane. It follows that the absolute velocity (that is, the velocity relative to the stationary frame) of that material point is $\langle \dot x_c, \dot y_c\rangle + \big\langle \sigma(\alpha), -\sigma'(\alpha)\big\rangle\dot\alpha$, that is, $\big\langle \dot x_c + \sigma(\alpha)\dot\alpha, \dot y_c - \sigma'(\alpha)\dot\alpha\big\rangle$. But since the ellipse rolls without slipping, that absolute velocity must be zero. We conclude that

$$\dot x_c + \sigma(\alpha)\dot\alpha = 0, \quad \dot y_c - \sigma'(\alpha)\dot\alpha = 0, \tag{7}$$

The kinetic and potential energies of the ellipse are

$$T = \frac{1}{2}m(\dot x_c^2 + \dot y_c^2) + \frac{1}{2}J\dot\alpha^2, \qquad V = mgy_c,$$

$J = \frac{1}{4}m(a^2 + b^2)$ is the moment of inertia about the axis perpendicular to the ellipse's surface and going through its center, m is the mass, and g is the acceleration due to gravity.

[2]Here and elsewhere, a dot superimposed on a symbol indicates the time derivative.

In view of equations (7), the Lagrangian $L = T - V$ of the ellipse may be expressed solely in terms of the generalized coordinate α. We obtain:

(8)

$$\ddot{\alpha} = \frac{-2(a^2 - b^2)\sin 2\alpha}{a^2b^2 + 5a^4\sin^2\alpha + 5b^4\cos^2\alpha}\left[\frac{a^2b^2\dot{\alpha}^2}{a^2\sin^2\alpha + b^2\cos^2\alpha} + g\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}\right].$$

The three differential equations in (7) and (8) may be solved along with suitable initial conditions to completely determine the ellipse's motion.

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