Feynman Diagrams
The scattering matrix in coordinates and momentum representation

Mathematical methods for *particle physics* was one of the weak spots in the *Physics* package. There existed a *FeynmanDiagrams* command, but its capabilities were too minimal. People working in the area asked for more functionality. These diagrams are the cornerstone of calculations in particle physics (collisions involving from the electron to the Higgs boson), for example at the CERN. As an introduction for people curious, not working in the area, see "Why Feynman Diagrams are so important".

This post is thus about a new development in *Physics*: a **full rewriting** of the *FeynmanDiagram*s command, now including a myriad of new capabilities (mainly a. b. and c. in the Introduction), reversing the previous status of things entirely. This is work in collaboration with Davide Polvara from Durham University, Centre for Particle Theory.

The complexity of this material is high, so the introduction to the presentation below is as brief as it can get, emphasizing the examples instead. This material is reproducible in Maple 2019.2 after installing the Physics Updates, v.598 or higher.

At the end it is attached the worksheet corresponding to this presentation, as well as the new *FeynmanDiagrams* help page with all the explanatory details.

Introduction

A scattering matrix S relates the initial and final states, $|\, i \,\rangle$ and $|\, f \,\rangle$, of an interacting system. In an 4-dimensional spacetime with coordinates X, S can be written as:

$$S = T\left(e^{i \int L(X)\, dX^4}\right)$$

where i is the imaginary unit and L is the *interaction* Lagrangian, written in terms of quantum fields depending on the spacetime coordinates X. The T symbol means *time-ordered*. For the terminology used in this page, see for instance *chapter IV*, *"The Scattering Matrix"*, of ref.[1] Bogoliubov, N.N., and Shirkov, D.V. **Quantum Fields**.

This exponential can be expanded as

$$S = 1 + S_1 + S_2 + S_3 + \ldots$$

where

$$S_n = \frac{i^n}{n!} \int \ldots \int T\left(L\left(X_1\right), \ldots, L\left(X_n\right)\right) dX_1^{\,4} \ldots dX_n^{\,4}$$

and $T\left(L\left(X_1\right), \ldots, L\left(X_n\right)\right)$ is the *time-ordered product* of n interaction Lagrangians evaluated at different points. The S matrix formulation is at the core of perturbative approaches in relativistic Quantum Field Theory.

In connection, the FeynmanDiagrams command has been rewritten entirely for Maple 2020. In brief, the new functionality includes computing:

a. The expansion $S = 1 + S_1 + S_2 + S_3 + \ldots$ in *coordinates representation* up to arbitrary order (the limitation is now only your hardware)

b. The S-matrix element $\langle\, f \,|\, S \,|\, i \,\rangle$ in *momentum representation* up to arbitrary order for given number of loops and initial and final particles (the contents of the $|\, i \,\rangle$ and $|\, f \,\rangle$ states); optionally, also the transition probability density, constructed using the square of the scattering matrix element $|\langle\, f \,|\, S \,|\, i \,\rangle|^2$, as shown in formula **(13)** of sec. 21.1 of ref.[1].

c. The Feynman diagrams (drawings) related to the different terms of the expansion of S or of its matrix elements $\langle\, f \,|\, S \,|\, i \,\rangle$.

Interaction Lagrangians involving derivatives of fields, typically appearing in non-Abelian gauge theories, are also handled, and several options are provided enabling restricting the outcome in different ways, regarding the incoming and outgoing particles, the number of loops, vertices or external legs, the propagators and normal products, or whether to compute tadpoles and 1-particle reducible terms.

Examples

For illustration purposes set three coordinate systems, and set ϕ to represent a quantum operator

> $with(Physics)$:

> $Setup(mathematicalnotation = true, coordinates = [X, Y, Z], quantumoperators = \phi)$

Systems of spacetime coordinates are: $\{X = (x1, x2, x3, x4), Y = (y1, y2, y3, y4), Z = (z1, z2, z3, z4)\}$

$$[coordinatesystems = \{X, Y, Z\}, mathematicalnotation = true, quantumoperators = \{\phi\}] \quad (1.1)$$

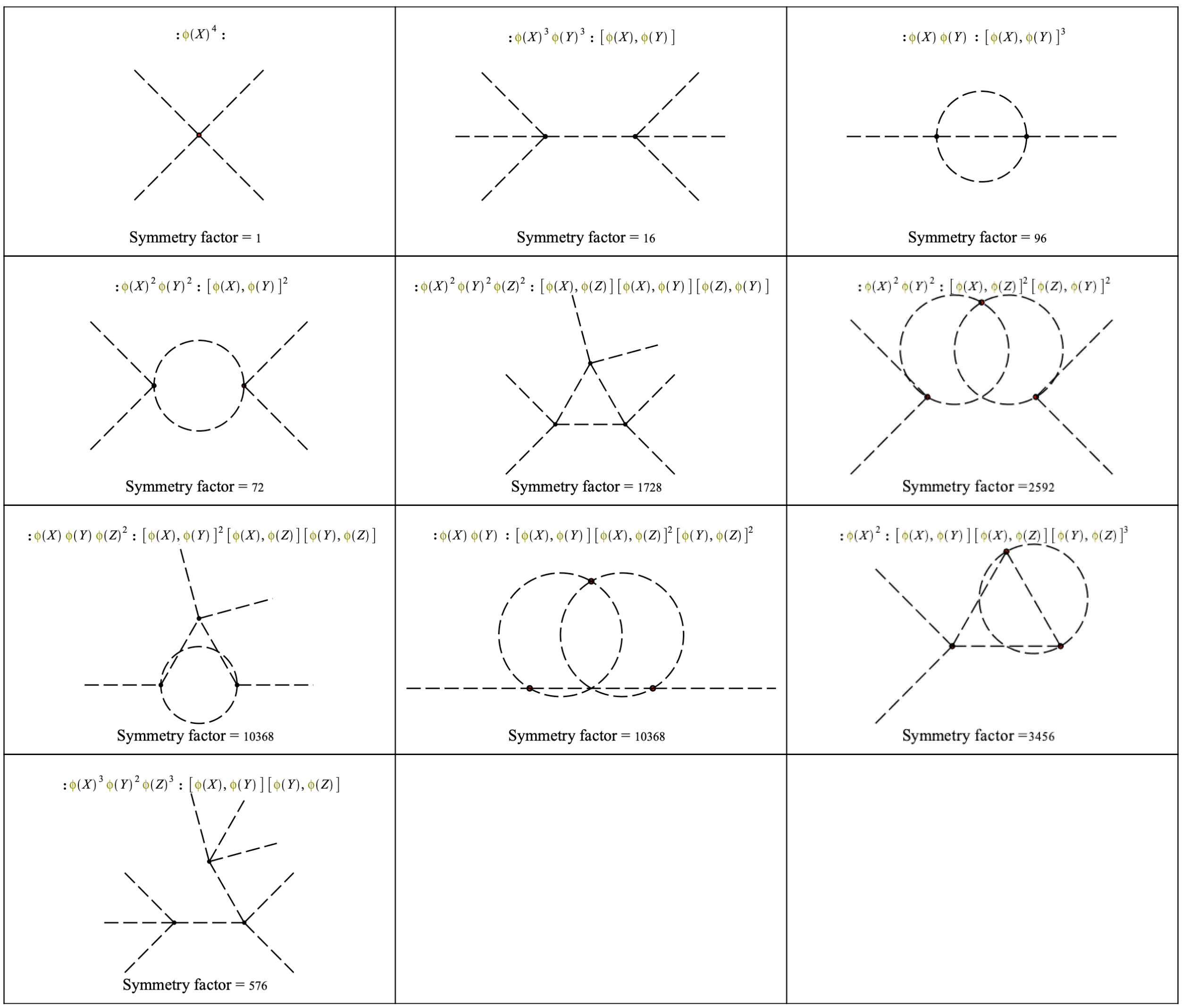
Let L be the interaction Lagrangian

> $L := \lambda\, \phi(X)^4$

$$L := \lambda\, \phi(X)^4 \quad (1.2)$$

The expansion of S in *coordinates representation,* computed by default up to $order = 3$ (you can change that using the option $order = n$), by definition containing all possible configurations of external legs, displaying the related Feynman Diagrams, is given by

> $S\big|_{order = 3} = FeynmanDiagrams(L, diagrams)$



$$S\big|_{order = 3} = 1 + I \int \lambda\, :\phi(X)^4: \,\mathrm{d}X^4 + \frac{I^2}{2!} \iint 16\, \lambda^2\, :\phi(X)^3\, \phi(Y)^3 : [\phi(X), \phi(Y)] \quad (1.3)$$

$$+ 96\,\lambda^2 :\phi(X)\,\phi(Y): [\phi(X), \phi(Y)]^3 + 72\,\lambda^2 :\phi(X)^2\,\phi(Y)^2: [\phi(X), \phi(Y)]^2\,\mathrm{d}X^4\mathrm{d}Y^4 + \frac{I^3}{3!}\iiint$$

$$1728\,\lambda^3 :\phi(X)^2\,\phi(Y)^2\,\phi(Z)^2: [\phi(X), \phi(Z)]\,[\phi(X), \phi(Y)]\,[\phi(Z), \phi(Y)]$$

$$+ 2592\,\lambda^3 :\phi(X)^2\,\phi(Y)^2: [\phi(X), \phi(Z)]^2[\phi(Z), \phi(Y)]^2 + 10368\,\lambda^3 :\phi(X)\,\phi(Y)\,\phi(Z)^2: [\phi(X),$$

$$\phi(Y)]^2\,[\phi(X), \phi(Z)]\,[\phi(Y), \phi(Z)] + 10368\,\lambda^3 :\phi(X)\,\phi(Y): [\phi(X), \phi(Y)]\,[\phi(X), \phi(Z)]^2\,[\phi(Y), \phi(Z)]^2$$

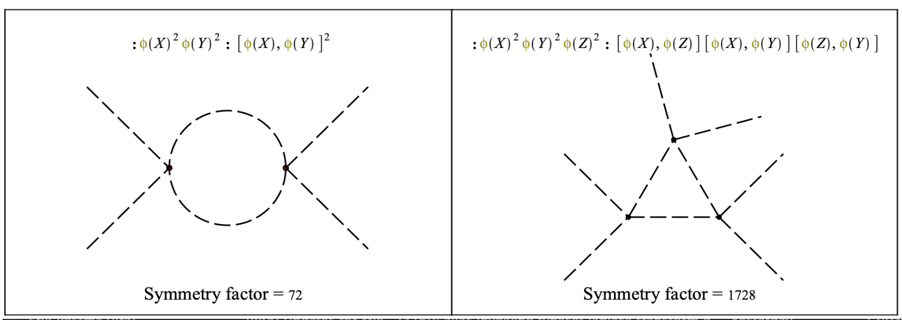
$$+ 3456\,\lambda^3 :\phi(X)^2: [\phi(X), \phi(Y)]\,[\phi(X), \phi(Z)]\,[\phi(Y), \phi(Z)]^3 + 576\,\lambda^3 :\phi(X)^3\,\phi(Y)^2\,\phi(Z)^3: [\phi(X),$$

$$\phi(Y)]\,[\phi(Y), \phi(Z)]\,\mathrm{d}X^4\mathrm{d}Y^4\mathrm{d}Z^4$$

The expansion of S in coordinates representation to a specific *order* shows in a compact way the topology of the underlying Feynman diagrams. Each *integral* is represented with a new command, FeynmanIntegral, that works both in *coordinates* and *momentum* representation. To each term of the integrands corresponds a diagram, and the correspondence is always clear from the symmetry factors.

In a typical situation, one wants to compute a specific term, or scattering process, instead of the S matrix up to some order with all possible configurations of external legs. For example, to compute only the terms of this result that correspond to diagrams with 1 loop use *numberofloops* = 1 (for *tree-level*, use *numerofloops* = 0)

> $S\Big|_{[order = 3,\ loops = 1]} = FeynmanDiagrams(L,\ numberofloops = 1,\ diagrams)$



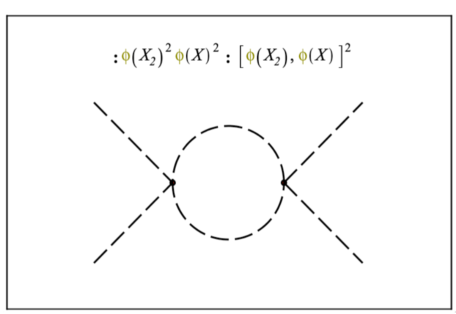
$$S\Big|_{[order = 3,\ loops = 1]} = \frac{I^2}{2!}\iint 72\,\lambda^2 :\phi(X)^2\,\phi(Y)^2: [\phi(X), \phi(Y)]^2\,\mathrm{d}X^4\mathrm{d}Y^4 + \frac{I^3}{3!}\iiint \quad \textbf{(1.4)}$$

$$1728\,\lambda^3 :\phi(X)^2\,\phi(Y)^2\,\phi(Z)^2: [\phi(X), \phi(Z)]\,[\phi(X), \phi(Y)]\,[\phi(Z), \phi(Y)]\,\mathrm{d}X^4\mathrm{d}Y^4\mathrm{d}Z^4$$

In the result above there are two terms, with 4 and 6 external legs respectively.

A scattering process with matrix element $\langle f \mid S \mid i \rangle$ in momentum representation, corresponding to the term with 4 external legs (symmetry factor = 72), could be any process where the total number of *incoming* + *outgoing* parties is equal to 4. For example, one with 2 incoming and 2 outgoing particles. The transition probability for that process is given by

> $\langle \phi, \phi \mid S \mid \phi, \phi \rangle = FeynmanDiagrams(L,\ incomingparticles = [\phi, \phi],\ outgoingparticles = [\phi, \phi],\ numberofloops = 1,\ diagrams)$



$$\langle \phi, \phi \mid S \mid \phi, \phi \rangle = \int \frac{9\,\lambda^2\,\delta(-P_3 - P_4 + P_1 + P_2)}{8\,\pi^6\,\sqrt{E_1 E_2 E_3 E_4}\,\left(p_2^{\,2} - m_\phi^{\,2} + I\,\epsilon\right)\left((-P_1 - P_2 - p_2)^2 - m_\phi^{\,2} + I\,\epsilon\right)}\,\mathrm{d}p_2^{\,4} + \int \quad \textbf{(1.5)}$$

$$\frac{9\,\lambda^2\,\delta\left(-P_3-P_4+P_1+P_2\right)}{8\,\pi^6\,\sqrt{E_1 E_2 E_3 E_4}\,\left(p_2^2-m_\phi^2+I\,\epsilon\right)\left(\left(-P_1+P_3-p_2\right)^2-m_\phi^2+I\,\epsilon\right)}\,dp_2^4+\int\frac{9\,\lambda^2\,\delta\left(-P_3-P_4+P_1+P_2\right)}{8\,\pi^6\,\sqrt{E_1 E_2 E_3 E_4}\,\left(p_2^2-m_\phi^2+I\,\epsilon\right)\left(\left(-P_1+P_4-p_2\right)^2-m_\phi^2+I\,\epsilon\right)}\,dp_2^4$$

When computing in *momentum representation*, only the topology of the corresponding Feynman diagrams is shown (i.e. the diagrams associated to the corresponding Feynman integral in *coordinates representation*).
The transition matrix element $\langle f \mid S \mid i \rangle$ is related to the transition probability density dw (formula **(13)** of sec. 21.1 of ref. [1]) by

$$dw=(2\,\pi)^{3\,s-4}\,n_1\,\ldots\,n_s\,\left|F\left(p_i,p_f\right)\right|^2\,\delta\left(\left(\sum_{i=1}^{s}p_i\right)-\left(\sum_{f=1}^{r}p_f\right)\right)\,d^3p_1\,\ldots\,d^3p_r$$

where $n_1\,\ldots\,n_s$ represent the particle densities of each of the s particles in the initial state $\mid i \rangle$, the δ (Dirac) is the expected singular factor due to the conservation of the energy-momentum and the *amplitude* $F\left(p_i,p_f\right)$ is related to $\langle f \mid S \mid i \rangle$ via

$$\langle f \mid S \mid i \rangle=F\left(p_i,p_f\right)\,\delta\left(\left(\sum_{i=1}^{s}p_i\right)-\left(\sum_{f=1}^{r}p_f\right)\right)$$

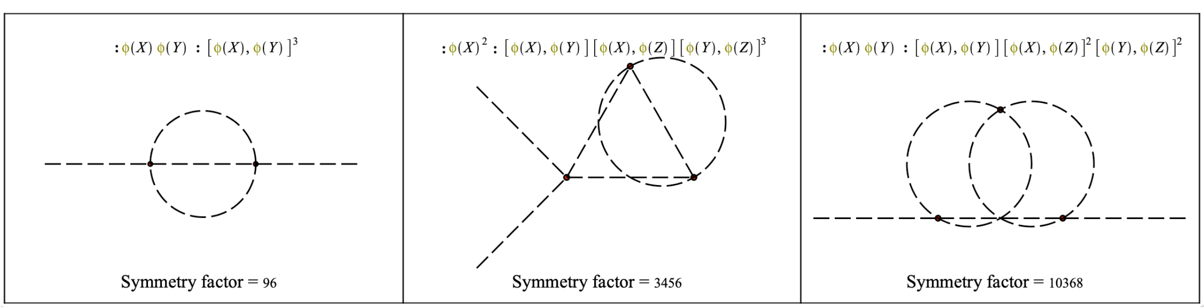
To directly get the *probability density* dw instead of $\langle f \mid S \mid i \rangle$ use the option *output* = *probabilitydensity*

> *FeynmanDiagrams*(*L*, *incomingparticles* = [φ, φ], *outgoingparticles* = [φ, φ], *numberofloops* = 1, *output* = *probabilitydensity*)

$$4\,\pi^2\,\prod_{i=1}^{2}n_i\,|F|^2\,\delta\left(-P_3-P_4+P_1+P_2\right)\prod_{f=1}^{2}\overrightarrow{dP_f}^{3}\ \textbf{where}\ F=\int\frac{9\,\lambda^2}{8\,\pi^6\,\sqrt{E_1 E_2 E_3 E_4}\,\left(p_2^2-m_\phi^2+I\,\epsilon\right)\left(\left(-P_1-P_2-p_2\right)^2-m_\phi^2+I\,\epsilon\right)}\,dp_2^4+\int\frac{9\,\lambda^2}{8\,\pi^6\,\sqrt{E_1 E_2 E_3 E_4}\,\left(p_2^2-m_\phi^2+I\,\epsilon\right)\left(\left(-P_1+P_3-p_2\right)^2-m_\phi^2+I\,\epsilon\right)}\,dp_2^4+\int\frac{9\,\lambda^2}{8\,\pi^6\,\sqrt{E_1 E_2 E_3 E_4}\,\left(p_2^2-m_\phi^2+I\,\epsilon\right)\left(\left(-P_1+P_4-p_2\right)^2-m_\phi^2+I\,\epsilon\right)}\,dp_2^4 \qquad \textbf{(1.6)}$$

In practice, the most common computations involve processes with 2 or 4 external legs. To restrict the expansion of the scattering matrix in *coordinates representation* to that kind of processes use the *numberofexternallegs* option. For example, the following computes the expansion of S up to *order* = 3, restricting the outcome to the terms corresponding to diagrams with only 2 *external legs*

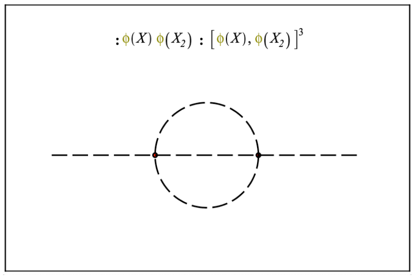
> $S\big|_{[order=3,\ legs=2]}$ = *FeynmanDiagrams*(*L*, *numberofexternallegs* = 2, *diagrams*)



$$S\Big|_{[order=3,\ legs=2]} = \frac{I^2}{2!}\iint 96\,\lambda^2 :\phi(X)\,\phi(Y): [\phi(X),\phi(Y)]^3\, dX^4 dY^4 + \frac{I^3}{3!}\iiint 3456\,\lambda^3 :\phi(X)^2: [\phi(X),\phi(Y)]\,[\phi(X),\phi(Z)]\,[\phi(Y),\phi(Z)]^3 + 10368\,\lambda^3 :\phi(X)\,\phi(Y): [\phi(X),\phi(Y)]\,[\phi(X),\phi(Z)]^2\,[\phi(Y),\phi(Z)]^2\, dX^4 dY^4 dZ^4 \tag{1.7}$$

This result shows two Feynman integrals, with respectively 2 and 3 loops, the second integral with two terms. The transition probability density in momentum representation for a process related to the first integral (1 term with symmetry factor = 96) is then

> *FeynmanDiagrams*(*L*, *incomingparticles* = [ϕ], *outgoingparticles* = [ϕ], *numberofloops* = 2, *diagrams*, *output* = *probabilitydensity*)



$$\frac{\prod_{i=1}^{1} n_i\, |F|^2\, \delta\left(-P_2 + P_1\right) \prod_{f=1}^{1} \overrightarrow{dP_f}^3}{2\,\pi} \quad \textbf{where } F = \iint \frac{\frac{3\,I}{8}\,\lambda^2}{\pi^7 \sqrt{E_1 E_2}\left(p_2^{\,2} - m_\phi^{\,2} + I\,\epsilon\right)\left(p_3^{\,2} - m_\phi^{\,2} + I\,\epsilon\right)\left(\left(-P_1 - p_2 - p_3\right)^2 - m_\phi^{\,2} + I\,\epsilon\right)}\, dp_2^{\,4}\, dp_3^{\,4} \tag{1.8}$$

In the above, for readability, the contracted spacetime indices in the square of momenta entering the amplitude *F* (as denominators of propagators) are implicit. To make those indices explicit, use the option *putindicesinsquareofmomentum*

> *F* = *FeynmanDiagrams*(*L*, *incoming* = [ϕ], *outgoing* = [ϕ], *numberofloops* = 2, *indices*)

* *Partial match of 'indices' against keyword 'putindicesinsquareofmomentum'*

$$F = \iint \left(\frac{3\,I}{8}\,\lambda^2\,\delta\left(-P_2^{\ \kappa} + P_1^{\ \kappa}\right)\right) \Big/ \left(\pi^7 \sqrt{E_1 E_2}\left(p_{2_\mu} p_2^{\ \mu} - m_\phi^{\,2} + I\,\epsilon\right)\left(p_{3_\nu} p_3^{\ \nu} - m_\phi^{\,2} + I\,\epsilon\right)\left(\left(-P_{1_\beta} - p_{2_\beta} - p_{3_\beta}\right)\left(-P_1^{\ \beta} - p_2^{\ \beta} - p_3^{\ \beta}\right) - m_\phi^{\,2} + I\,\epsilon\right)\right) dp_2^{\,4}\, dp_3^{\,4} \tag{1.9}$$

This computation can also be performed to higher orders. For example, with 3 loops, in *coordinates* and *momentum* representations, corresponding to the other two terms and diagrams in **??**

> $S_3\Big|_{[legs=2,\ loops=3]}$ = *FeynmanDiagrams*(*L*, *legs* = 2, *loops* = 3)

* *Partial match of 'legs' against keyword 'numberoflegs'*

* *Partial match of 'loops' against keyword 'numberofloops'*

$$S_3\Big|_{[legs=2,\ loops=3]} = \frac{I^3}{3!}\iiint 3456\,\lambda^3 :\phi(X)^2: [\phi(X),\phi(Y)]\,[\phi(X),\phi(Z)]\,[\phi(Y),\phi(Z)]^3 + 10368\,\lambda^3 :\phi(X)\,\phi(Y): [\phi(X),\phi(Y)]\,[\phi(X),\phi(Z)]^2\,[\phi(Y),\phi(Z)]^2\, dX^4 dY^4 dZ^4 \tag{1.10}$$

A corresponding S-matrix element in momentum representation:

> $\left\langle \phi \,\middle|\, S_3 \,\middle|\, \phi \right\rangle\Big|_{loops=3}$ = *FeynmanDiagrams*(*L*, *incomingparticles* = [ϕ], *outgoingparticles* = [ϕ], *numberofloops* = 3)

(1.11)

$$\langle \phi \mid S_3 \mid \phi \rangle \Big|_{loops = 3} = \int\int\int \left(9\,\lambda^3\,\delta(-P_2 + P_1)\right) \Big/ \Big(32\,\pi^{11}\,\sqrt{E_1 E_2}\,(p_3^2 - m_\phi^2 + I\,\epsilon)\,(p_4^2 - m_\phi^2 + I\,\epsilon)\,(p_5^2 - m_\phi^2 + I\,\epsilon)\,((-p_3 - p_4 - p_5)^2 - m_\phi^2 + I\,\epsilon)\,((-P_1 + P_2 + p_3 + p_4 + p_5)^2 - m_\phi^2 + I\,\epsilon)\Big)\, dp_3^4\, dp_4^4\, dp_5^4 + 2\int\int\int \left(9\,\lambda^3\,\delta(-P_2 + P_1)\right) \Big/ \Big(32\,\pi^{11}\,\sqrt{E_1 E_2}\,(p_3^2 - m_\phi^2 + I\,\epsilon)\,(p_4^2 - m_\phi^2 + I\,\epsilon)\,(p_5^2 - m_\phi^2 + I\,\epsilon)\,((-p_3 - p_4 - p_5)^2 - m_\phi^2 + I\,\epsilon)\,((-P_1 + p_4 + p_5)^2 - m_\phi^2 + I\,\epsilon)\Big)\, dp_3^4\, dp_4^4\, dp_5^4 + \int\int \frac{\lambda\,\delta(-P_2 + P_1) \displaystyle\int \frac{576\,\lambda^2}{(p_2^2 - m_\phi^2 + I\,\epsilon)\,((-p_2 - p_4 - p_5)^2 - m_\phi^2 + I\,\epsilon)}\, dp_2^4}{2048\,\pi^{11}\,\sqrt{E_1 E_2}\,(p_4^2 - m_\phi^2 + I\,\epsilon)\,(p_5^2 - m_\phi^2 + I\,\epsilon)\,((-P_1 + p_4 + p_5)^2 - m_\phi^2 + I\,\epsilon)}\, dp_4^4\, dp_5^4 \tag{1.11}$$

Consider the interaction Lagrangian of Quantum Electrodynamics (QED). To formula this problem on the worksheet, start defining the vector field A_μ.

> $Define(A[\mu])$

Defined objects with tensor properties

$$\{A_\mu, \gamma_\mu, P_{1_\mu}, P_{2_\mu}, \sigma_\mu, \partial_\mu, g_{\mu,\nu}, p_{1_\mu}, p_{2_\mu}, p_{3_\mu}, p_{4_\mu}, p_{5_\mu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu, Y_\mu, Z_\mu\} \tag{1.12}$$

Set lowercase Latin letters from i to s to represent spinor indices (you can change this setting according to your preference, see Setup), also the (anticommutative) spinor field will be represented by ψ, so set ψ as an *anticommutativeprefix*, and set A and ψ as quantum operators

> $Setup(spinorindices = lowercaselatin\_is,\ anticommutativeprefix = \psi,\ op = \{A, \psi\})$

** Partial match of 'op' against keyword 'quantumoperators'*

$$[anticommutativeprefix = \{\psi\},\ quantumoperators = \{A, \phi, \psi\},\ spinorindices = lowercaselatin\_is] \tag{1.13}$$

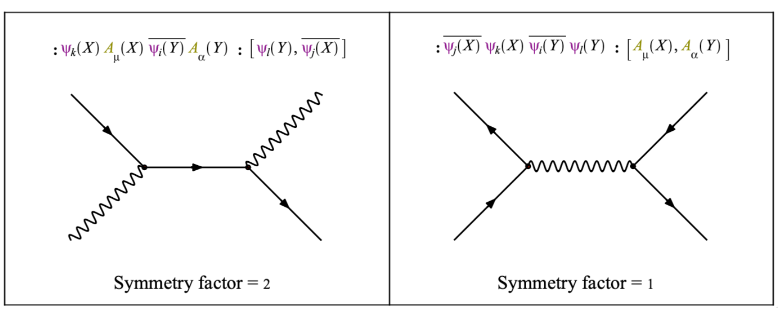
The matrix indices of the Dirac matrices are written explicitly and use conjugate to represent the Dirac conjugate $\overline{\psi_j}$

> $L_{QED} := \alpha\ conjugate(\psi[j](X))\ Dgamma[\mu][j, k]\ \psi[k](X)\ A[\mu](X)$

$$L_{QED} := \alpha\,\overline{\psi_j(X)}\,\psi_k(X)\,A_\mu(X)\,(\gamma^\mu)_{j,k} \tag{1.14}$$

Compute S_2, only the terms with 4 external legs, and display the *diagrams*: all the corresponding graphs have no loops

> $S_2\Big|_{legs = 4} = FeynmanDiagrams(L_{QED},\ numberofvertices = 2,\ numberoflegs = 4,\ diagrams)$

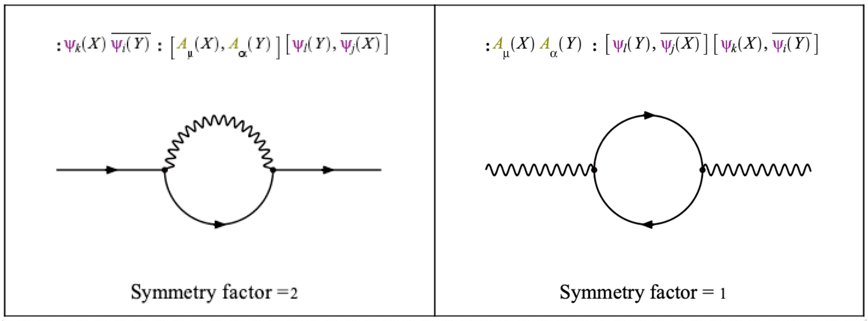


$$S_2\Big|_{legs = 4} = \frac{I^2}{2!}\int\int -2\,\alpha^2\,(\gamma^\mu)_{j,k}\,(\gamma^\alpha)_{i,l}\ :\psi_k(X)\,A_\mu(X)\,\overline{\psi_i(Y)}\,A_\alpha(Y):\ [\psi_l(Y), \overline{\psi_j(X)}] \tag{1.15}$$

$$+\ \alpha^2 \left(\gamma^{\mu}\right)_{j,k} \left(\gamma^{\alpha}\right)_{i,l} :\overline{\psi_j(X)}\ \psi_k(X)\ \overline{\psi_i(Y)}\ \psi_l(Y): \left[A_{\mu}(X), A_{\alpha}(Y)\right] \mathrm{d}X^4 \mathrm{d}Y^4$$

The same computation but with only 2 external legs results in the diagrams with 1 loop that correspond to the self-energy of the electron and the photon (page 218 of ref.[1])

> $S_2\Big|_{legs=2} = FeynmanDiagrams\left(L_{QED},\ numberofvertices = 2,\ numberoflegs = 2,\ diagrams\right)$



$$S_2\Big|_{legs=2} = \frac{I^2}{2!}\iint -2\,\alpha^2 \left(\gamma^{\mu}\right)_{j,k} \left(\gamma^{\alpha}\right)_{i,l} :\psi_k(X)\ \overline{\psi_i(Y)}: \left[A_{\mu}(X), A_{\alpha}(Y)\right]\left[\psi_l(Y), \overline{\psi_j(X)}\right] \tag{1.16}$$

$$-\ \alpha^2 \left(\gamma^{\mu}\right)_{j,k} \left(\gamma^{\alpha}\right)_{i,l} :A_{\mu}(X)\ A_{\alpha}(Y): \left[\psi_l(Y), \overline{\psi_j(X)}\right]\left[\psi_k(X), \overline{\psi_i(Y)}\right] \mathrm{d}X^4 \mathrm{d}Y^4$$

where the diagram with two spinor legs is the electron self-energy. To restrict the output furthermore, for example getting only the self-energy of the photon, you can specify the *normal products* you want:

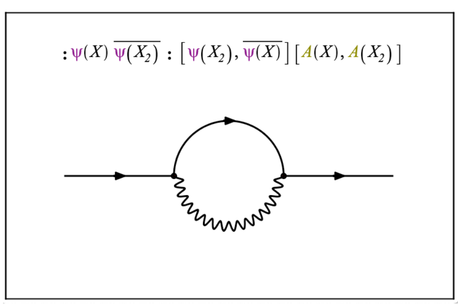
> $S_2\Big|_{[legs=2,\ products=\ :A^2:]} = FeynmanDiagrams\left(L_{QED},\ numberofvertices = 2,\ numberoflegs = 2,\ normalproduct = \ :A^2:\right)$

** Partial match of 'normalproduct' against keyword 'normalproducts'*

$$S_2\Big|_{[legs=2,\ products=\ :A^2:]} = \frac{I^2}{2!}\iint \alpha^2 \left(\gamma^{\mu}\right)_{j,k} \left(\gamma^{\alpha}\right)_{i,l} :A_{\mu}(X)\ A_{\alpha}(Y): \left[\overline{\psi_j(X)}, \psi_l(Y)\right]\left[\psi_k(X), \overline{\psi_i(Y)}\right] \mathrm{d}X^4 \mathrm{d}Y^4 \tag{1.17}$$

The corresponding S-matrix elements in momentum representation

> $\langle\ \psi \mid S \mid \psi\ \rangle = FeynmanDiagrams\left(L_{QED},\ incomingparticles = [\psi],\ outgoing = [\psi],\ numberofloops = 1,\ diagrams\right)$



$$\langle\ \psi \mid S \mid \psi\ \rangle = -\int \frac{1}{8\,\pi^3 \left(p_2^{\ 2} - m_A^{\ 2} + I\,\epsilon\right)\left(\left(P_1 + p_2\right)^2 - m_{\psi}^{\ 2} + I\,\epsilon\right)} \left(\left(\boldsymbol{u}_{\psi}\right)_i\left(\overrightarrow{P_1}\right)\overline{\left(\boldsymbol{u}_{\psi}\right)_l\left(\overrightarrow{P_2}\right)}\left(-g_{\alpha,\nu}\right.\right. \tag{1.18}$$

$$+\frac{p_{2_\nu} p_{2_\alpha}}{m_A^2}\Bigg) \alpha^2 \left(\gamma^\alpha\right)_{l,m} \left(\gamma^\nu\right)_{n,i} \left(\left(P_{1_\beta} + p_{2_\beta}\right)\left(\gamma^\beta\right)_{m,n} + m_\psi \delta_{m,n}\right) \delta\left(-P_2 + P_1\right)\Bigg] \mathrm{d}p_2^{\,4}$$

In this result we see $\boldsymbol{u}_\psi$ spinor (see ref.[2]), and the propagator of the field A_μ with a mass m_A. To indicate that this field is *massless* use

> *Setup*(*massless* = *A*)

** Partial match of 'massless' against keyword 'masslessfields'*

$$[masslessfields = \{A\}] \tag{1.19}$$

Now the propagator for A_μ is the one of a massless vector field:

> *FeynmanDiagrams*(L_{QED}, *incoming* = [psi], *outgoing* = [psi], *numberofloops* = 1)

$$-\int -\frac{\left(\boldsymbol{u}_\psi\right)_i\left(\vec{P}_1\right)\overline{\left(\boldsymbol{u}_\psi\right)_l\left(\vec{P}_2\right)}\, g_{\alpha,\nu}\, \alpha^2 \left(\gamma^\alpha\right)_{l,m}\left(\gamma^\nu\right)_{n,i}\left(\left(P_{1_\beta}+p_{2_\beta}\right)\left(\gamma^\beta\right)_{m,n} + m_\psi \delta_{m,n}\right)\delta\left(-P_2+P_1\right)}{8\,\pi^3\left(p_2^{\,2} + \mathrm{I}\,\epsilon\right)\left(\left(P_1+p_2\right)^2 - m_\psi^{\,2} + \mathrm{I}\,\epsilon\right)} \mathrm{d}p_2^{\,4} \tag{1.20}$$

The self-energy of the photon:

> ⟨ *A* | *S* | *A* ⟩ = *FeynmanDiagrams*(L_{QED}, *incomingparticles* = [*A*], *outgoing* = [*A*], *numberofloops* = 1)

$$\langle A \mid S \mid A \rangle = -\int \Bigg(\left(\boldsymbol{\varepsilon}_A\right)_\nu\left(\vec{P}_1\right)\overline{\left(\boldsymbol{\varepsilon}_A\right)_\alpha\left(\vec{P}_2\right)}\left(m_\psi \delta_{l,n} + p_{2_\beta}\left(\gamma^\beta\right)_{l,n}\right)\alpha^2\left(\gamma^\alpha\right)_{n,i}\left(\gamma^\nu\right)_{m,l}\left(\left(P_{1_\tau} + p_{2_\tau}\right)\left(\gamma^\tau\right)_{i,m} + m_\psi \delta_{i,m}\right)\delta\left(-P_2+P_1\right)\Bigg) \Big/ \left(16\,\pi^3\sqrt{E_1 E_2}\left(p_2^{\,2} - m_\psi^{\,2} + \mathrm{I}\,\epsilon\right)\left(\left(P_1+p_2\right)^2 - m_\psi^{\,2} + \mathrm{I}\,\epsilon\right)\right) \mathrm{d}p_2^{\,4} \tag{1.21}$$

where $\boldsymbol{\varepsilon}_A$ is the corresponding polarization vector.

When working with non-Abelian gauge fields, the interaction Lagrangian involves derivatives. FeynmanDiagrams can handle that kind of interaction in *momentum representation*. Consider for instance a Yang-Mills theory with a massless field $B_{\mu,a}$ where a is a SU2 index (see eq.(12) of sec. 19.4 of ref.[1]). The interaction Lagrangian can be entered as follows

> *Setup*(*su2indices* = *lowercaselatin_ah*, *massless* = *B*, *op* = *B*)

** Partial match of 'massless' against keyword 'masslessfields'*

** Partial match of 'op' against keyword 'quantumoperators'*

$$[masslessfields = \{A, B\},\ quantumoperators = \{A, B, \phi, \psi, \psi 1\},\ su2indices = lowercaselatin\_ah] \tag{1.22}$$

> *Define*(*B*[μ, *a*], *quiet*) :

> F_B[μ, ν, *a*] := *d*_[μ](*B*[ν, *a*](*X*)) − *d*_[ν](*B*[μ, *a*](*X*))

$$F_{B_{\mu,\nu,a}} := \partial_\mu\left(B_{\nu,a}(X)\right) - \partial_\nu\left(B_{\mu,a}(X)\right) \tag{1.23}$$

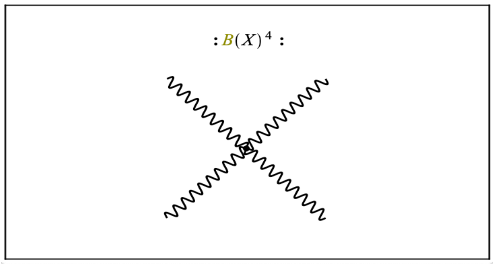
> $L := \frac{g}{2}$ *LeviCivita*[*a*, *b*, *c*] F_B[μ, ν, *a*] *B*[μ, *b*](*X*) *B*[ν, *c*](*X*) + $\frac{g^2}{4}$ *LeviCivita*[*a*, *b*, *c*] *LeviCivita*[*a*, *e*, *f*] *B*[μ, *b*](*X*) *B*[ν, *c*](*X*) *B*[μ, *e*](*X*) *B*[ν, *f*](*X*)

$$L := \frac{g\,\epsilon_{a,b,c}\left(\partial_\mu\left(B_{\nu,a}(X)\right) - \partial_\nu\left(B_{\mu,a}(X)\right)\right) B^{\mu}{}_{b}(X)\, B^{\nu}{}_{c}(X)}{2} + \frac{g^2\,\epsilon_{a,b,c}\,\epsilon_{a,e,f}\, B_{\mu,b}(X)\, B_{\nu,c}(X)\, B^{\mu}{}_{e}(X)\, B^{\nu}{}_{f}(X)}{4} \tag{1.24}$$

The transition probability density at tree-level for a process with two incoming and two outgoing B particles is given by

> *FeynmanDiagrams*(*L*, *incomingparticles* = [*B*, *B*], *outgoingparticles* = [*B*, *B*], *numberofloops* = 0, *output* = *probabilitydensity*, *factor*, *diagrams*)

* *Partial match of 'factor' against keyword 'factortreelevel'* **(1.25)**



$$4\,\pi^2 \prod_{i=1}^{2} n_i\, |F|^2\, \delta\left(-P_3^{\ \sigma} - P_4^{\ \sigma} + P_1^{\ \sigma} + P_2^{\ \sigma}\right) \prod_{f=1}^{2} \overrightarrow{dP_f}^{3} \textbf{ where } F \qquad \textbf{(1.26)}$$

$$= \frac{1}{\pi^2 \sqrt{E_1 E_2 E_3 E_4}} \Bigg(\Bigg(\frac{1}{8\left(\left(-P_{1_\chi} - P_{2_\chi}\right)\left(-P_1^{\ \chi} - P_2^{\ \chi}\right) + \mathrm{I}\,\epsilon\right)} \Big(\mathrm{I}\,\epsilon_{a1,\,a3,\,h} \Big(\Big(\left(-P_1^{\ \kappa} - P_2^{\ \kappa} - P_4^{\ \kappa}\right)$$

$$g^{\lambda,\tau} + \left(P_1^{\ \lambda} + P_2^{\ \lambda} + P_3^{\ \lambda}\right) g^{\kappa,\tau} - g^{\kappa,\lambda}\left(P_3^{\ \tau} - P_4^{\ \tau}\right)\Big)\Big)\,\epsilon_{a2,\,d,\,g} \Big(\Big(\left(P_1^{\ \beta} + \frac{P_2^{\ \beta}}{2}\right) g^{\alpha,\sigma} + \Big($$

$$-\frac{P_1^{\ \sigma}}{2} + \frac{P_2^{\ \sigma}}{2}\Big) g^{\alpha,\beta} - \frac{g^{\beta,\sigma}\left(P_1^{\ \alpha} + 2\,P_2^{\ \alpha}\right)}{2}\Big) g_{\sigma,\tau}\, \delta_{a2,\,a3}\Big)$$

$$- \frac{1}{16\left(\left(-P_{1_\chi} + P_{3_\chi}\right)\left(-P_1^{\ \chi} + P_3^{\ \chi}\right) + \mathrm{I}\,\epsilon\right)} \Big(\mathrm{I}\,\Big(\Big(\left(-P_1^{\ \beta} + P_3^{\ \beta} - P_4^{\ \beta}\right) g^{\lambda,\tau} + \Big(P_1^{\ \lambda} - P_2^{\ \lambda} -$$

$$P_3^{\ \lambda}\Big) g^{\beta,\tau} + g^{\beta,\lambda}\left(P_2^{\ \tau} + P_4^{\ \tau}\right)\Big)\Big)\,\epsilon_{a1,\,a3,\,g} \Big(\Big(\left(P_1^{\ \sigma} + P_3^{\ \sigma}\right) g^{\alpha,\kappa} + \left(-2\,P_1^{\ \kappa} + P_3^{\ \kappa}\right) g^{\alpha,\sigma} +$$

$$g^{\kappa,\sigma}\left(P_1^{\ \alpha} - 2\,P_3^{\ \alpha}\right)\Big)\Big)\,\epsilon_{a2,\,d,\,h}\, g_{\sigma,\tau}\, \delta_{a2,\,a3}\Big) - \frac{1}{16\left(\left(-P_{1_\chi} + P_{4_\chi}\right)\left(-P_1^{\ \chi} + P_4^{\ \chi}\right) + \mathrm{I}\,\epsilon\right)} \Big(\mathrm{I}\,\Big(\Big(-$$

$$P_1^{\ \beta} - P_3^{\ \beta} + P_4^{\ \beta}\Big) g^{\kappa,\tau} + \left(P_1^{\ \kappa} - P_2^{\ \kappa} - P_4^{\ \kappa}\right) g^{\beta,\tau} + g^{\beta,\kappa}\left(P_2^{\ \tau} + P_3^{\ \tau}\right)\Big)\Big)\,\epsilon_{a3,\,g,\,h} \Big(\Big(P_1^{\ \sigma}$$

$$+ P_4^{\sigma}\Big) g^{\alpha,\lambda} + \Big(P_1^{\alpha} - 2 P_4^{\alpha}\Big) g^{\lambda,\sigma} - 2 g^{\alpha,\sigma}\left(P_1^{\lambda} - \frac{P_4^{\lambda}}{2}\right)\Bigg)\epsilon_{a1,a2,d}\, g_{\sigma,\tau}\, \delta_{a2,a3}\Bigg)$$

$$-\frac{1}{16}\Bigg(\mathrm{I}\Big(\delta_{g,h}\,\delta_{a1,d}\big(g^{\alpha,\beta} g^{\kappa,\lambda} + g^{\alpha,\kappa} g^{\beta,\lambda} - 2 g^{\beta,\kappa} g^{\alpha,\lambda}\big) + \delta_{d,h}\big(g^{\alpha,\beta} g^{\kappa,\lambda} - 2 g^{\alpha,\kappa} g^{\beta,\lambda} + g^{\beta,\kappa} g^{\alpha,\lambda}\big)\delta_{a1,g} - 2\left(g^{\alpha,\beta} g^{\kappa,\lambda} - \frac{g^{\beta,\kappa} g^{\alpha,\lambda}}{2} - \frac{g^{\alpha,\kappa} g^{\beta,\lambda}}{2}\right)\delta_{d,g}\,\delta_{a1,h}\Big)\Bigg) g^2\, \overline{(\boldsymbol{\varepsilon}_B)_{\kappa,h}(\vec{P}_3)}\; \overline{(\boldsymbol{\varepsilon}_B)_{\lambda,a1}(\vec{P}_4)}\; (\boldsymbol{\varepsilon}_B)_{\alpha,d}(\vec{P}_1)\; (\boldsymbol{\varepsilon}_B)_{\beta,g}(\vec{P}_2)\Bigg)$$

To simplify the repeated indices, us the option *simplifytensorindices*. To check the indices entering a result like this one use Check; there are no free indices, and regarding the repeated indices:

> *Check*((**1.26**), *all*)

The repeated indices per term are: $[\{\ldots\}, \{\ldots\}, \ldots]$, *the free indices are:* $\{\ldots\}$

$$[\{a1, a2, a3, \alpha, \beta, \chi, d, g, h, \kappa, \lambda, \sigma, \tau\}], \varnothing \tag{1.27}$$

This process can be computed with 1 or more loops, in which case the number of terms increases significantly. As another interesting non-Abelian model, consider the interaction Lagrangian of the electro-weak part of the Standard Model

> *Coordinates*(*clear*, *Z*)

Unaliasing $\{Z\}$ *previously defined as a system of spacetime coordinates* **(1.28)**

> *Setup*(*quantumoperators* = {*W*, *Z*})

$$[quantumoperators = \{A, B, W, Z, \phi, \psi, \psi 1\}] \tag{1.29}$$

> *Define*(*W*[μ], *Z*[μ])

Defined objects with tensor properties

$$\{A_{\mu}, B_{\mu,a}, \gamma_{\mu}, P_{1_{\mu}}, P_{2_{\mu}}, P_{3_{\alpha}}, P_{4_{\alpha}}, \sigma_{\mu}, W_{\mu}, Z_{\mu}, \partial_{\mu}, g_{\mu,\nu}, p_{1_{\mu}}, p_{2_{\mu}}, p_{3_{\mu}}, p_{4_{\mu}}, p_{5_{\mu}}, \psi_j, \epsilon_{\alpha,\beta,\mu,\nu}, X_{\mu}, Y_{\mu}\} \tag{1.30}$$

> *CompactDisplay*((*W*, *Z*)(*X*))

W(*X*) *will now be displayed as W*

Z(*X*) *will now be displayed as Z* **(1.31)**

> $F_W[\mu, \nu] := d\_[\mu](W[\nu](X)) - d\_[\nu](W[\mu](X))$

$$F_{W_{\mu,\nu}} := \partial_{\mu}(W_{\nu}) - \partial_{\nu}(W_{\mu}) \tag{1.32}$$

> $F_Z[\mu, \nu] := d\_[\mu](Z[\nu](X)) - d\_[\nu](Z[\mu](X))$

$$F_{Z_{\mu,\nu}} := \partial_{\mu}(Z_{\nu}) - \partial_{\nu}(Z_{\mu}) \tag{1.33}$$

> $L_{WZ} := \mathrm{I}\, g \cos(\theta_w)\, \big(\big(Dagger(F_W[\mu,\nu])\, W[\mu](X) - Dagger(W[\mu](X))\, F_W[\mu,\nu]\big)\, Z[\nu](X) + W[\nu](X)\, Dagger(W[\mu](X))\, F_Z[\mu,\nu]\big)$

$$L_{WZ} := \mathrm{I}\, g \cos(\theta_w)\Big(\Big(\Big(\partial_{\mu}(W_{\nu}^{\dagger}) - \partial_{\nu}(W_{\mu}^{\dagger})\Big) W^{\mu} - W_{\mu}^{\dagger}\Big(\partial^{\mu}(W_{\nu}) - \partial_{\nu}(W^{\mu})\Big)\Big) Z^{\nu} + W_{\nu} W_{\mu}^{\dagger}\Big(\partial^{\mu}(Z^{\nu}) - \partial^{\nu}(Z^{\mu})\Big)\Big) \tag{1.34}$$

This interaction Lagrangian contains six different terms. The S-matrix element for the tree-level process with two incoming and two outgoing *W* particles is shown in the help page for FeynmanDiagrams.

>

References

[1] Bogoliubov, N.N., and Shirkov, D.V. **Quantum Fields**. Benjamin Cummings, 1982.
[2] Weinberg, S., **The Quantum Theory Of Fields**. Cambridge University Press, 2005.