This is an interesting exercise, the computation of the Liénard–Wiechert potentials describing the classical electromagnetic field of a moving electric point charge, a problem of a 3rd year undergrad course in Electrodynamics. The calculation is nontrivial and is performed below using the Physics package, following the presentation in [1] (Landau & Lifshitz "*The classical theory of fields*"). I have not seen this calculation performed on a computer algebra worksheet before. Thus, this also showcases the more advanced level of symbolic problems that can currently be tackled on a Maple worksheet. At the end, the corresponding document is linked and with it the computation below can be reproduced. There is also a link to a corresponding PDF file with all the sections open.

Moving charges: The retarded and Liénard-Wiechert potentials, electric and magnetic fields $\vec{E}$ and $\vec{H}$

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Generally speaking, determining the electric and magnetic fields of a distribution of charges involves determining the potentials φ and $\vec{A}$, followed by determining the fields $\vec{E}$ and $\vec{H}$ from

$$\vec{E} = -\frac{1}{c}\left(\frac{\partial}{\partial t}\vec{A}\right) - \nabla\varphi, \qquad \vec{H} = \nabla\times\vec{A}$$

In turn, the formulation of the equations for φ and $\vec{A}$ is simple: they follow from the 4D second pair of Maxwell equations, in tensor notation

$$\partial_k\left(F^{i,k}\right) = -\frac{4\pi}{c}j^i$$

where $F^{i,k}$ is the electromagnetic field tensor and j^i is the 4D current. After imposing the Lorentz condition

$$\partial_i\left(A^i\right) = 0, \quad \text{i.e.} \quad \frac{1}{c}\left(\frac{\partial}{\partial t}\varphi\right) + \nabla\cdot\vec{A} = 0$$

we get

$$\partial_k\left(\partial^k\left(A^i\right)\right) = \frac{4\pi}{c}j^i$$

which in 3D form results in

$$\nabla^2\vec{A} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\vec{A}\right) = -\frac{4\pi}{c}\vec{j}$$

$$\nabla^2\varphi - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\varphi\right) = -\frac{4\pi}{c}\rho$$

where $\vec{j}$ is the current and ρ is the charge density.

Following the presentation shown in [1] (Landau and Lifshitz, "***The classical theory of fields***", sec. 62

and 63), below we solve these equations for φ and $\vec{A}$ resulting in the so-called *retarded potentials*, then recompute these fields as produced by a charge moving along a given trajectory $\vec{r} = r_0(t)$ - the so-called Liénard-Wiechert potentials - finally computing an explicit form for the corresponding $\vec{E}$ and $\vec{H}$.

While the computation of the generic retarded potentials is, in principle, simple, obtaining their form for a charge moving along a given trajectory $\vec{r} = r_0(t)$, and from there the form of the fields $\vec{E}$ and $\vec{H}$ shown in Landau's book, involves nontrivial algebraic manipulations. The presentation below thus also shows a technique to map onto the computer the manipulations typically done with paper and pencil for these problems. To reproduce the contents below, the Maplesoft Physics Updates v.1252 or newer is required.

```
with(Physics) :
Setup(coordinates = Cartesian);
with(Vectors) :
```

Systems of spacetime coordinates are: $\{X = (x, y, z, t)\}$

$$[coordinatesystems = \{X\}] \tag{1}$$

The retarded potentials ϕ and $\vec{A}$

The equations which determine the scalar and vector potentials of an arbitrary electromagnetic field are input as

```
CompactDisplay((φ, ρ, A_, j_)(X))
```

$\varphi(x, y, z, t)$ *will now be displayed as* φ

$\rho(x, y, z, t)$ *will now be displayed as* ρ

$\vec{A}(x, y, z, t)$ *will now be displayed as* $\vec{A}$

$\vec{j}(x, y, z, t)$ *will now be displayed as* $\vec{j}$ **(2)**

```
%Laplacian(φ(X)) − 1/c^2·diff(φ(X), t, t) = −4·π·ρ(X)
```

$$\nabla^2 \varphi - \frac{\varphi_{t,t}}{c^2} = -4\pi\rho \tag{3}$$

```
%Laplacian(A_(X)) − 1/c^2·diff(A_(X), t, t) = −4·π·j_(X)
```

$$\nabla^2 \vec{A} - \frac{\vec{A}_{t,t}}{c^2} = -4\pi\vec{j} \tag{4}$$

The solutions to these inhomogeneous equations are computed as the sum of the solutions for the equations without right-hand side plus a particular solution to the equation with right-hand side.

Computing the solution to the equations for $\varphi(X)$ and $\vec{A}(X)$

Following [1], to find a particular solution, divide the whole space into infinitely small regions and determine the field produced by a charge located in one of these volume elements. By linearity of the field equations, the actual field will be the sum of the fields produced by all such elements. The charge de in a given volume element is, generally speaking, a function of the time. If we choose the origin of coordinates in the volume element under consideration, then the charge density is $\rho = de(t)\delta(r)$, where r is the radial coordinate, the distance from the origin. For $\varphi(X)$ we then have

$subs(\mathrm{rho}(X) = de(t)\cdot \mathrm{Dirac}([x, y, z]), value(\mathbf{(3)}))$

$$\varphi_{x,x} + \varphi_{y,y} + \varphi_{z,z} - \frac{\varphi_{t,t}}{c^2} = -4\pi\, de(t)\, \delta^{(3)}([x, y, z]) \qquad \mathbf{(5)}$$

Switching to polar coordinates to take advantage of the spherical symmetry,

$tr := [x, y, z] =\sim ChangeCoordinates([x, y, z], spherical)$

$$tr := [x = r\sin(\theta)\cos(\phi), y = r\sin(\theta)\sin(\phi), z = r\cos(\theta)] \qquad \mathbf{(6)}$$

$PDEtools\text{:-}dchange(tr, \mathbf{(5)}, [r, \theta, \phi, t], simplify)$

$$\frac{1}{\sin(\theta)^2 r^2 c^2}\Big(\big(-c^2\cos(\theta)^2 + c^2\big)\big(\varphi_{\theta,\theta}\big) + \big(-c^2 r^2\cos(\theta)^2 + r^2 c^2\big)\big(\varphi_{r,r}\big) + \big(r^2\cos(\theta)^2 - r^2\big)\big(\varphi_{t,t}\big) + c^2\Big(\varphi_{\phi,\phi} + \big(-2r\cos(\theta)^2 + 2r\big)\big(\varphi_r\big) + \sin(\theta)\cos(\theta)\big(\varphi_\theta\big)\Big)\Big) = -4\pi\, de(t)\, \delta^{(3)}([r\cos(\theta), r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi)]) \qquad \mathbf{(7)}$$

The spherical symmetry implies that

$(eval(\mathbf{(7)}, [\varphi(r, \theta, \phi, t) = \varphi(r, t), \mathrm{Dirac}([r\cos(\theta), r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi)]) = \mathrm{Dirac}(r)]))$

$$\frac{1}{\sin(\theta)^2 r^2 c^2}\Big(\big(-c^2 r^2\cos(\theta)^2 + r^2 c^2\big)\big(\varphi_{r,r}\big) + \big(r^2\cos(\theta)^2 - r^2\big)\big(\varphi_{t,t}\big) + c^2\big(-2r\cos(\theta)^2 + 2r\big)\big(\varphi_r\big)\Big) = -4\pi\, de(t)\, \delta(r) \qquad \mathbf{(8)}$$

This equation can be tackled *as is* using pdsolve resulting in the sum of advanced $\left(t + \frac{r}{c}\right)$ and retarded $\left(t - \frac{r}{c}\right)$ functions

$pdsolve(\mathbf{(8)})$

$$\varphi(r, t) = \frac{\_F1\left(t + \frac{r}{c}\right)c + 2\,\_F2\left(t - \frac{r}{c}\right)}{2r} \qquad \mathbf{(9)}$$

Since we only want a particular solution satisfying **(8)** at the origin and vanishing at infinity - only one, we choose the retarded function, suffices.

$eval(\mathbf{(9)}, \_F1 = 0)$

$$\varphi(r, t) = \frac{\_F2\left(t - \frac{r}{c}\right)}{r} \qquad \mathbf{(10)}$$

To specify $\_F2\left(t - \frac{r}{c}\right)$ such that $\varphi(r, t)$ behaves as desired at the origin and at infinity, it helps to simplify the PDE **(8)** in order to analyze the problem

$expand(simplify((8)))$

$$\varphi_{r,r} + \frac{2(\varphi_r)}{r} - \frac{\varphi_{t,t}}{c^2} = -4\pi\, de(t)\, \delta(r) \qquad \textbf{(11)}$$

By construction, the solution computed, $\frac{\_F2\left(t - \frac{r}{c}\right)}{r}$ satisfies **(11)** when $r \neq 0$. For the adjustment of $\_F2$ when $r \to 0$, in that case the derivative with respect to r is much greater than the derivative with respect to t, so that the term $-\frac{\varphi_{t,t}}{c^2}$ can be neglected and **(11)** becomes

$\%Laplacian(\varphi(r, t)) = rhs((11))$

$$\nabla^2 \varphi(r, t) = -4\pi\, de(t)\, \delta(r) \qquad \textbf{(12)}$$

Now, $\nabla^2\left(\frac{1}{r}\right) = -4 \cdot \pi \cdot \delta(r)$, and the solution $\varphi(r, t) = \frac{\_F2\left(t - \frac{r}{c}\right)}{r}$ already has a factor $\frac{1}{r}$ so we can choose $\_F2\left(t - \frac{r}{c}\right)$ such that, in the limiting case when $r \to 0$, it is a function of t only, and a function of r with no singularities otherwise. By inspection in **(12)** such a function can be the element of charge $de\left(t - \frac{r}{c}\right)$, so the solution of **(11)** that behaves as desired at the origin, also at infinity where $de\Big|_{r \to \infty} \to 0$, consists of taking $\_F2\left(t - \frac{r}{c}\right) = de\left(t - \frac{r}{c}\right)$

$subs\left(\_F2\left(t - \frac{r}{c}\right) = de\left(t - \frac{r}{c}\right), (10)\right)$

$$\varphi(r, t) = \frac{de\left(t - \frac{r}{c}\right)}{r} \qquad \textbf{(13)}$$

which is the form of the potential shown in Landau's book after its equation (62.7). From this result, one gets the solution to **??** for an arbitrary distribution of charges $\rho(x, y, z, t)$ by writing

$$de(t, r) = \rho(t, r)\ dV\ ,$$

then

$$r = |\vec{r} - \vec{r}\,'|,$$

where $\vec{r} - \vec{r}\,'$ is the distance from the volume element dV' to the point where the potential $\varphi(\vec{r}, t)$ is evaluated, and integrating over the whole space

$$\varphi(\vec{r}, t) = \int \frac{1}{|\vec{r} - \vec{r}\,'|} \cdot \rho\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right) \mathrm{d}V'$$

This expression of the potential $\varphi(\vec{r}, t)$, shown in Landau's book as (62.9), is called the *retarded potential*. The same approach results in the *retarded vector potential* $\vec{A}(\vec{r}, t)$:

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int \frac{1}{|\vec{r} - \vec{r}'|} \cdot \vec{j}\left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}\right) \mathrm{d}V'$$

where $\vec{j}$ is the electric current density.

The Liénard-Wiechert potentials of a charge moving along $\vec{r} = \vec{r}_0(t)$

From **(13)**, the potential at the point $X = (x, y, z, t)$ is determined by the charge $e\left(t - \frac{r}{c}\right)$, i.e. by the position of the charge e at the earlier time

$$t' = t - \frac{\|\vec{R}\|}{c}$$

The quantity $\|\vec{R}\|$ is the 3D distance from ***the position of the charge at the time*** t' to the 3D point of observation (x, y, z). In the previous section, the charge was located at the origin and at rest, so $\|\vec{R}\| = r$, the radial coordinate. If the charge is moving, say on a path $\vec{r}_0(t)$, we have

$$\vec{R} = \vec{r} - \vec{r}_0(t')$$

From **(13)** $\equiv \varphi(r, t) = \dfrac{de\left(t - \frac{r}{c}\right)}{r}$ and the definition of t' above, the potential $\varphi(r, t)$ of a moving charge can be written as

$$\varphi(r, t) = \frac{e}{\|\vec{R}\|} = \frac{e}{c\cdot(t - t')}$$

When the charge is at rest, in the Lorentz gauge we are working, the vector potential is $\vec{A} = 0$. When the charge is moving, the form of $\vec{A}$ can be found searching for a solution to $\nabla^2\vec{A} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\vec{A}\right) = -\frac{4\pi}{c}\vec{j}$ that gives $\vec{A} = 0$ when $\vec{v} = 0$. Following [1], this solution can be written as

$$A^{\alpha} = \frac{e\, u^{\alpha}}{R_{\beta}\, u^{\beta}}$$

where u^{μ} is the four velocity of the charge, $R^{\mu} = r^{\mu} - r_0^{\mu} = \left[\vec{r} - \vec{r}', c(t - t')\right]$.

Without showing the intermediate steps, [1] presents the three dimensional vectorial form of these potentials φ and $\vec{A}$ as

$$\varphi = \frac{e}{R - \left(\frac{\vec{v}}{c}\right)\cdot\vec{R}}, \quad \vec{A} = \frac{e\,\vec{v}}{c\left(R - \left(\frac{\vec{v}}{c}\right)\cdot\vec{R}\right)}$$

Computing the vectorial form of the Liénard-Wiechert potentials

To compute these two expressions, start with some definitions. Instead of the prime notation, we will use the simpler Maple notation of literal subscripts, so $t_0 \equiv t'$, $x_0 \equiv x'$, ... With this notation, the first equation, relating the difference $t - t'$ to the norm $\|\vec{R}\|$ is

$$t_0 = t - \frac{Norm(R\_)}{c}$$

$$t_0 = t - \frac{\|\vec{R}\|}{c} \quad \textbf{(14)}$$

Define now the tensors of this problem in terms of their covariant components

$$Define\left(u_\mu = \left[-\frac{\gamma v_x}{c}, -\frac{\gamma v_y}{c}, -\frac{\gamma v_z}{c}, \gamma\right], r_\mu = [-x, -y, -z, c\,t], r_{0_\mu} = [-x_0, -y_0, -z_0, c\,t_0], R_\mu = r_\mu - r_{0_\mu}, quiet\right):$$

where γ represents $\frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}}$. To check any of these definitions,

$$u[\sim]$$

$$u^\mu = \begin{bmatrix} \frac{\gamma v_x}{c} & \frac{\gamma v_y}{c} & \frac{\gamma v_z}{c} & \gamma \end{bmatrix} \quad \textbf{(15)}$$

$$R[definition]$$

$$R_\mu = r_\mu - r_{0_\mu} \quad \textbf{(16)}$$

$$R[\sim]$$

$$R^\mu = \begin{bmatrix} x - x_0 & y - y_0 & z - z_0 & c\,(t - t_0) \end{bmatrix} \quad \textbf{(17)}$$

Define now the 4-potential A_α in terms of these quantities as indicated in Landau's book

$$A[\text{alpha}] = e \cdot \frac{u[\text{alpha}]}{R[\text{mu}] \cdot u[\text{mu}]}$$

$$A_\alpha = \frac{e\,u_\alpha}{R_\mu u^\mu} \quad \textbf{(18)}$$

$$Define(\textbf{(18)}, quiet):$$

The scalar potential $\varphi(t, x, y, z)$ is now given by the 0^{th} component of A^α

$$\varphi = A[\sim 0]$$

$$\varphi = \frac{e\,c}{c^2\,t - c^2\,t_0 - v_x\,x + v_x\,x_0 - v_y\,y + v_y\,y_0 - v_z\,z + v_z\,z_0} \quad \textbf{(19)}$$

Next is to remove the retarded time using **(14)** $\equiv$ $t_0 = t - \frac{\|\vec{R}\|}{c}$, and to introduce the vectorial form

$\vec{R}\cdot\vec{v}$. For that purpose introduce first Latin lower case letters as *space* indices

Setup(*spaceindices* = *lowercaselatin*)

$$[spaceindices = lowercaselatin] \tag{20}$$

$\vec{R}\cdot\vec{v}$ can now be expressed in terms of the 3D tensor part of R_α and u_α using space indices

$$-\frac{R[j]\cdot u[j]c}{\text{gamma}} = R\_ . v\_$$

$$-\frac{c\,R_j\,u^j}{\gamma} = \vec{R}\cdot\vec{v} \tag{21}$$

SumOverRepeatedIndices(**(21)**)

$$v_x\,x - v_x\,x_0 + v_y\,y - v_y\,y_0 + v_z\,z - v_z\,z_0 = \vec{R}\cdot\vec{v} \tag{22}$$

The expression **(19)** for the potential φ is simplified to

simplify(**(19)**, {**(14)**, **(22)**})

$$\varphi = \frac{e\,c}{c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}} \tag{23}$$

The exact expression (63.5) can be obtained by multiplying and dividing by c the right-hand side

$$lhs(\mathbf{(23)}) = \frac{subs\left(Norm(R\_) = \frac{Norm(R\_)}{c},\, v\_ = \frac{v\_}{c},\, rhs(\mathbf{(23)})\right)}{c}$$

$$\varphi = \frac{e}{\|\vec{R}\| - \vec{R}\cdot\left(\frac{\vec{v}}{c}\right)} \tag{24}$$

For the vector potential $\vec{A}$, we have

$A\_ = A[\sim 1]\cdot\_i + A[\sim 2]\cdot\_j + A[\sim 3]\cdot\_k$

$$\vec{A} = \frac{e\,v_x\,\hat{i}}{c^2\,t - c^2\,t_0 - v_x\,x + v_x\,x_0 - v_y\,y + v_y\,y_0 - v_z\,z + v_z\,z_0} + \frac{e\,v_y\,\hat{j}}{c^2\,t - c^2\,t_0 - v_x\,x + v_x\,x_0 - v_y\,y + v_y\,y_0 - v_z\,z + v_z\,z_0} + \frac{e\,v_z\,\hat{k}}{c^2\,t - c^2\,t_0 - v_x\,x + v_x\,x_0 - v_y\,y + v_y\,y_0 - v_z\,z + v_z\,z_0} \tag{25}$$

This expression can be simplified in the same way, expressing the retarded time t_0 in terms of $\|\vec{R}\|$ using **(14)**, and introducing $\vec{R}\cdot\vec{v}$ using **(22)**

simplify(**(25)**, {**(14)**, **(22)**})

$$\vec{A} = \frac{e\left(\hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z\right)}{c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}} \tag{26}$$

Substituting

$subs\left(v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k} = \vec{v}, \mathbf{(26)}\right)$

$$\vec{A} = \frac{e\,\vec{v}}{c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}} \tag{27}$$

To obtain (63.5) shown in the textbook, multiply and divide by c

$subs\left(Norm(R\_) = \frac{Norm(R\_)}{c}, v\_ = \frac{v\_}{c}, \textbf{(27)}\right)$

$$\vec{A} = \frac{e\,\vec{v}}{\left(\|\vec{R}\| - \vec{R}\cdot\left(\frac{\vec{v}}{c}\right)\right)c} \tag{28}$$

The electric and magnetic fields $\vec{E}$ and $\vec{H}$ of a charge moving along $\vec{r} = \vec{r}_0(t)$

The electric and magnetic fields at a point (x, y, z, t) are calculated from the potentials φ and $\vec{A}$ through the formulas

$$\vec{E}(x, y, z, t) = -\frac{1}{c}\left(\frac{\partial}{\partial t}\vec{A}(x, y, z, t)\right) - \nabla\varphi(x, y, z, t), \qquad \vec{H}(x, y, z, t) = \nabla\times\vec{A}(x, y, z, t)$$

where, for the case of a charge moving on a path $\vec{r}_0(t)$, these potentials were calculated in the previous section as **(24)** and **(18)**

$$\varphi(x, y, z, t) = \frac{e}{\|\vec{R}\| - \vec{R}\cdot\left(\frac{\vec{v}}{c}\right)}$$

$$\vec{A}(x, y, z, t) = \frac{e\,\vec{v}}{c\cdot\left(\|\vec{R}\| - \vec{R}\cdot\left(\frac{\vec{v}}{c}\right)\right)}$$

These two expressions, however, depend on the time only through the retarded time t_0. This dependence is within $\vec{R} = \vec{r}(x, y, z) - \vec{r}_0\left(t_0(x, y, z, t)\right)$ and through the velocity of the charge $\vec{v}\left(t_0(x, y, z, t)\right)$. So, before performing the differentiations, this dependence on $t_0(x, y, z, t)$ must be taken into account.

$CompactDisplay\left(r\_(x, y, z), \left(\left(E\_, H\_, t_0\right)(x, y, z, t)\right)\right)$

$\vec{r}(x, y, z)$ *will now be displayed as* $\vec{r}$

$\vec{E}(x, y, z, t)$ *will now be displayed as* $\vec{E}$

$\vec{H}(x, y, z, t)$ *will now be displayed as* $\vec{H}$

$t_0(x, y, z, t)$ *will now be displayed as* t_0 **(29)**

$R\_ = r\_(x, y, z) - r_{0\_}\left(t_0(x, y, z, t)\right), v\_ = v\_\left(t_0(x, y, z, t)\right)$

$$\vec{R} = \vec{r} - \vec{r}_0\left(t_0\right), \vec{v} = \vec{v}\left(t_0\right) \tag{30}$$

The Electric field $\vec{E} = -\frac{1}{c}\left(\frac{\partial}{\partial t}\vec{A}\right) - \nabla\varphi$

Computation of $\nabla\varphi$

Start by introducing the dependency on the retarded time $\vec{R} = \vec{r} - \vec{r}_0\left(t_0(x, y, z, t)\right), \vec{v} = \vec{v}\left(t_0(x, y, z, t)\right)$

$subs\left(\mathbf{(30)}, \varphi = \varphi(x, y, z, t), \mathbf{(24)}\right)$

$$\varphi = \frac{e}{\left\|\vec{r} - \vec{r}_0(t_0)\right\| - \frac{\left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \vec{v}(t_0)}{c}} \tag{31}$$

$\%Gradient(lhs(\mathbf{(31)})) = Gradient(rhs(\mathbf{(31)}))$

$$\nabla\varphi = -\left(e\,c\left(\left(\left(\vec{r}_x - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_x})\right) \cdot \left(\vec{r} - \vec{r}_0(t_0)\right)\right)c + \left(-\left(\vec{r}_x - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_x})\right) \cdot \vec{v}(t_0) - (t_{0_x})\left(\left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \mathrm{D}(\vec{v})(t_0)\right)\right)\left\|\vec{r} - \vec{r}_0(t_0)\right\|\right)\hat{i}\right) \Big/ \left(\left(\left\|\vec{r} - \vec{r}_0(t_0)\right\| c - \left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \vec{v}(t_0)\right)^2 \left\|\vec{r} - \vec{r}_0(t_0)\right\|\right) - \left(e\,c\left(\left(\left(\vec{r}_y - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_y})\right) \cdot \left(\vec{r} - \vec{r}_0(t_0)\right)\right)c + \left(-\left(\vec{r}_y - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_y})\right) \cdot \vec{v}(t_0) - (t_{0_y})\left(\left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \mathrm{D}(\vec{v})(t_0)\right)\right)\left\|\vec{r} - \vec{r}_0(t_0)\right\|\right)\hat{j}\right) \Big/ \left(\left(\left\|\vec{r} - \vec{r}_0(t_0)\right\| c - \left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \vec{v}(t_0)\right)^2 \left\|\vec{r} - \vec{r}_0(t_0)\right\|\right) - \left(e\,c\left(\left(\left(\vec{r}_z - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_z})\right) \cdot \left(\vec{r} - \vec{r}_0(t_0)\right)\right)c + \left(-\left(\vec{r}_z - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_z})\right) \cdot \vec{v}(t_0) - (t_{0_z})\left(\left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \mathrm{D}(\vec{v})(t_0)\right)\right)\left\|\vec{r} - \vec{r}_0(t_0)\right\|\right)\hat{k}\right) \Big/ \left(\left(\left\|\vec{r} - \vec{r}_0(t_0)\right\| c - \left(\vec{r} - \vec{r}_0(t_0)\right) \cdot \vec{v}(t_0)\right)^2 \left\|\vec{r} - \vec{r}_0(t_0)\right\|\right) \tag{32}$$

The expression $\mathrm{D}(r_{0\_})(t_0(t, x, y, z))$ is the velocity $\vec{v}(t_0(t, x, y, z))$ and the derivatives $\frac{\partial\vec{r}}{\partial x}, \frac{\partial\vec{r}}{\partial y}, \frac{\partial\vec{r}}{\partial z}$ are the unit vectors

$\mathrm{D}(\vec{r}_0)(t_0(x, y, z, t)) = \vec{v}, \mathrm{D}(\vec{v})(t_0(x, y, z, t)) = a\_$

$$\mathrm{D}(\vec{r}_0)(t_0) = \vec{v}, \mathrm{D}(\vec{v})(t_0) = \vec{a} \tag{33}$$

$seq\left(diff\left(\vec{r}(x, y, z) = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z, q\right), q = [x, y, z]\right)$

$$\vec{r}_x = \hat{i}, \vec{r}_y = \hat{j}, \vec{r}_z = \hat{k} \tag{34}$$

We also want to replace

$map(rhs = lhs, [\mathbf{(30)}])$

$$\left[\vec{r} - \vec{r}_0(t_0) = \vec{R}, \vec{v}(t_0) = \vec{v}\right] \tag{35}$$

Substituting all at once,

$subs(\mathbf{(33)}, \mathbf{(34)}, \mathbf{(35)}, \mathbf{(32)})$

$$\nabla\varphi = -\frac{e\,c\left(\left(\left(\hat{i} - \vec{v}\,(t_{0_x})\right) \cdot \vec{R}\right)c + \left(-\left(\hat{i} - \vec{v}\,(t_{0_x})\right) \cdot \vec{v} - (t_{0_x})\left(\vec{R} \cdot \vec{a}\right)\right)\|\vec{R}\|\right)\hat{i}}{\left(c\,\|\vec{R}\| - \vec{R} \cdot \vec{v}\right)^2 \|\vec{R}\|} - \frac{e\,c\left(\left(\left(\hat{j} - \vec{v}\,(t_{0_y})\right) \cdot \vec{R}\right)c + \left(-\left(\hat{j} - \vec{v}\,(t_{0_y})\right) \cdot \vec{v} - (t_{0_y})\left(\vec{R} \cdot \vec{a}\right)\right)\|\vec{R}\|\right)\hat{j}}{\left(c\,\|\vec{R}\| - \vec{R} \cdot \vec{v}\right)^2 \|\vec{R}\|} \tag{36}$$

$$- \frac{e\,c\left(\left(\left(\hat{k} - \vec{v}\left(t_{0_z}\right)\right)\cdot\vec{R}\right)c + \left(-\left(\hat{k} - \vec{v}\left(t_{0_z}\right)\right)\cdot\vec{v} - \left(t_{0_z}\right)\left(\vec{R}\cdot\vec{a}\right)\right)\|\vec{R}\|\right)\hat{k}}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^2\|\vec{R}\|}$$

Simplify(**(36)**)

$$\nabla\varphi = \frac{1}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^2\|\vec{R}\|}\left(c\left(\hat{i}\left(-\|\vec{R}\|\,\|\vec{v}\|^2 + c\left(\vec{R}\cdot\vec{v}\right) + \left(\vec{R}\cdot\vec{a}\right)\|\vec{R}\|\right)\left(t_{0_x}\right) + \hat{j}\left(-\|\vec{R}\|\,\|\vec{v}\|^2 + c\left(\vec{R}\cdot\vec{v}\right) + \left(\vec{R}\cdot\vec{a}\right)\|\vec{R}\|\right)\left(t_{0_y}\right) + \hat{k}\left(-\|\vec{R}\|\,\|\vec{v}\|^2 + c\left(\vec{R}\cdot\vec{v}\right) + \left(\vec{R}\cdot\vec{a}\right)\|\vec{R}\|\right)\left(t_{0_z}\right) - \vec{R}\,c + \|\vec{R}\|\,\vec{v}\right)e\right) \quad (37)$$

By eye this expression involves ∇t_0:

collect(*subs*($-\|\vec{R}\|\,\|\vec{v}\|^2 + c\left(\vec{R}\cdot\vec{v}\right) + \left(\vec{R}\cdot\vec{a}\right)\|\vec{R}\| = U$, **(37)**), U)

$$\nabla\varphi = \frac{c\left(\hat{i}\left(t_{0_x}\right) + \hat{k}\left(t_{0_z}\right) + \hat{j}\left(t_{0_y}\right)\right)e\,U}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^2\|\vec{R}\|} + \frac{c\left(\|\vec{R}\|\,\vec{v} - \vec{R}\,c\right)e}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^2\|\vec{R}\|} \quad (38)$$

So,

(*Gradient* = *%Gradient*) $\left(t_0(x, y, z, t)\right)$

$$\hat{i}\left(t_{0_x}\right) + \hat{k}\left(t_{0_z}\right) + \hat{j}\left(t_{0_y}\right) = \nabla t_0 \quad (39)$$

simplify(**(37)**, { **(39)** })

$$\nabla\varphi = \frac{c\,e\left(\left(c\left(\vec{R}\cdot\vec{v}\right) + \|\vec{R}\|\left(-\|\vec{v}\|^2 + \vec{R}\cdot\vec{a}\right)\right)\left(\nabla t_0\right) - \vec{R}\,c + \|\vec{R}\|\,\vec{v}\right)}{\|\vec{R}\|\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^2} \quad (40)$$

At this point we need to compute ∇t_0. Starting with the definition of retarded time

subs ($t_0 = t_0(X)$, **(30)**, **(14)**)

$$t_0 = t - \frac{\|\vec{r} - \vec{r}_0(t_0)\|}{c} \quad (41)$$

Before differentiating, check that we have the right functionality

show

$$t_0(X) = t - \frac{\|\vec{r}(x, y, z) - \vec{r}_0\left(t_0(X)\right)\|}{c} \quad (42)$$

Take the gradient

Nabla (**(41)**)

$$\hat{i}\left(t_{0_x}\right) + \hat{k}\left(t_{0_z}\right) + \hat{j}\left(t_{0_y}\right) = -\frac{\left(\left(\vec{r}_x - \mathrm{D}\left(\vec{r}_0\right)\left(t_0\right)\left(t_{0_x}\right)\right)\cdot\left(\vec{r} - \vec{r}_0\left(t_0\right)\right)\right)\hat{i}}{\|\vec{r} - \vec{r}_0\left(t_0\right)\|\,c} - \frac{\left(\left(\vec{r}_y - \mathrm{D}\left(\vec{r}_0\right)\left(t_0\right)\left(t_{0_y}\right)\right)\cdot\left(\vec{r} - \vec{r}_0\left(t_0\right)\right)\right)\hat{j}}{\|\vec{r} - \vec{r}_0\left(t_0\right)\|\,c} \quad (43)$$

$$- \frac{\left(\left(\vec{r}_z - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_z})\right) \cdot (\vec{r} - \vec{r}_0(t_0))\right) \hat{k}}{\|\vec{r} - \vec{r}_0(t_0)\|\, c}$$

Replace $\frac{\partial \vec{r}}{\partial x} = \hat{i}$, $\frac{\partial \vec{r}}{\partial y} = \hat{j}$, $\frac{\partial \vec{r}}{\partial z} = \hat{k}$ using **(34)**

subs(**(34)**, **(43)**)

$$\hat{i}\,(t_{0_x}) + \hat{k}\,(t_{0_z}) + \hat{j}\,(t_{0_y}) = - \frac{\left(\left(\hat{i} - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_x})\right) \cdot (\vec{r} - \vec{r}_0(t_0))\right) \hat{i}}{\|\vec{r} - \vec{r}_0(t_0)\|\, c} - \frac{\left(\left(\hat{j} - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_y})\right) \cdot (\vec{r} - \vec{r}_0(t_0))\right) \hat{j}}{\|\vec{r} - \vec{r}_0(t_0)\|\, c} - \frac{\left(\left(\hat{k} - \mathrm{D}(\vec{r}_0)(t_0)\,(t_{0_z})\right) \cdot (\vec{r} - \vec{r}_0(t_0))\right) \hat{k}}{\|\vec{r} - \vec{r}_0(t_0)\|\, c} \tag{44}$$

After having differentiated we can remove functionality and reintroduce ∇t_0 instead of the expanded form

subs(**(33)**, **(35)**, **(44)**)

$$\hat{i}\,(t_{0_x}) + \hat{k}\,(t_{0_z}) + \hat{j}\,(t_{0_y}) = - \frac{\left(\left(\hat{i} - \vec{v}\,(t_{0_x})\right) \cdot \vec{R}\right) \hat{i}}{\|\vec{R}\|\, c} - \frac{\left(\left(\hat{j} - \vec{v}\,(t_{0_y})\right) \cdot \vec{R}\right) \hat{j}}{\|\vec{R}\|\, c} - \frac{\left(\left(\hat{k} - \vec{v}\,(t_{0_z})\right) \cdot \vec{R}\right) \hat{k}}{\|\vec{R}\|\, c} \tag{45}$$

Simplify(**(45)**)

$$\hat{i}\,(t_{0_x}) + \hat{k}\,(t_{0_z}) + \hat{j}\,(t_{0_y}) = \frac{(\vec{R} \cdot \vec{v})\,(t_{0_x})\,\hat{i} + (\vec{R} \cdot \vec{v})\,(t_{0_z})\,\hat{k} + (\vec{R} \cdot \vec{v})\,(t_{0_y})\,\hat{j} - \vec{R}}{c\,\|\vec{R}\|} \tag{46}$$

simplify(**(46)**, {**(39)**})

$$\nabla t_0 = \frac{(\vec{R} \cdot \vec{v})\,(\nabla t_0) - \vec{R}}{c\,\|\vec{R}\|} \tag{47}$$

isolate(**(47)**, *%Gradient*($t_0(x, y, z, t)$))

$$\nabla t_0 = \frac{\vec{R}}{-c\,\|\vec{R}\| + \vec{R} \cdot \vec{v}} \tag{48}$$

Inserting this result into the expression **(40)** for $\nabla \varphi$ and simplifying

lhs(**(40)**) = *simplify*(*subs*(**(48)**, *rhs*(**(40)**)))

$$\nabla \varphi = - \frac{c\, e \left(- \|\vec{v}\|^2 \vec{R} - \|\vec{R}\|\, c\, \vec{v} + \vec{R}\, c^2 + (\vec{R} \cdot \vec{a})\, \vec{R} + (\vec{R} \cdot \vec{v})\, \vec{v}\right)}{\left(c\,\|\vec{R}\| - \vec{R} \cdot \vec{v}\right)^3} \tag{49}$$

Computation of

$$\frac{\partial \vec{A}}{\partial t}$$

This computation proceeds the same way as the one done for $\nabla \varphi$. Start from the form **(28)** of the retarded vector potential $\vec{A}$ derived in section 1, introduce there the dependency on the retarded time t_0 through **(30)** $\equiv \vec{R} = \vec{r} - \vec{r}_0(t_0(x, y, z, t)), \vec{v} = \vec{v}(t_0(x, y, z, t))$, then differentiate with respect to t and simplify the result

subs(**(30)**, *A_* = *A_*(*x*, *y*, *z*, *t*), **(28)**)

$$\vec{A} = \frac{e\,\vec{v}(t_0)}{\left(\|\vec{r} - \vec{r}_0(t_0)\| - \frac{(\vec{r} - \vec{r}_0(t_0)) \cdot \vec{v}(t_0)}{c}\right) c} \tag{50}$$

diff(**(50)**, *t*)

$$\vec{A}_t = \frac{e\,\vec{v}(t_0)\,(t_{0_t})\,(\mathrm{D}(\vec{r}_0)(t_0) \cdot (\vec{r} - \vec{r}_0(t_0)))}{\left(\|\vec{r} - \vec{r}_0(t_0)\| - \frac{(\vec{r} - \vec{r}_0(t_0)) \cdot \vec{v}(t_0)}{c}\right)^2 c\,\|\vec{r} - \vec{r}_0(t_0)\|} - \frac{e\,\vec{v}(t_0)\,(t_{0_t})\,(\mathrm{D}(\vec{r}_0)(t_0) \cdot \vec{v}(t_0))}{\left(\|\vec{r} - \vec{r}_0(t_0)\| - \frac{(\vec{r} - \vec{r}_0(t_0)) \cdot \vec{v}(t_0)}{c}\right)^2 c^2} + \frac{e\,\vec{v}(t_0)\,(t_{0_t})\,((\vec{r} - \vec{r}_0(t_0)) \cdot \mathrm{D}(\vec{v})(t_0))}{\left(\|\vec{r} - \vec{r}_0(t_0)\| - \frac{(\vec{r} - \vec{r}_0(t_0)) \cdot \vec{v}(t_0)}{c}\right)^2 c^2} + \frac{e\,\mathrm{D}(\vec{v})(t_0)\,(t_{0_t})}{\left(\|\vec{r} - \vec{r}_0(t_0)\| - \frac{(\vec{r} - \vec{r}_0(t_0)) \cdot \vec{v}(t_0)}{c}\right) c} \tag{51}$$

The expression $\mathrm{D}(r_{0\_})(t_0(t, x, y, z))$ is the velocity $\vec{v}(t_0(t, x, y, z))$, $\mathrm{D}(\vec{v})(t_0(x, y, z, t))$ is the acceleration $a\_$ and We also want to replace **(30)**

$\mathrm{D}(\vec{r}_0)(t_0(x, y, z, t)) = \vec{v}, \mathrm{D}(\vec{v})(t_0(x, y, z, t)) = a\_$, *map*(*rhs* = *lhs*, [**(30)**])

$$\mathrm{D}(\vec{r}_0)(t_0) = \vec{v}, \mathrm{D}(\vec{v})(t_0) = \vec{a}, \left[\vec{r} - \vec{r}_0(t_0) = \vec{R}, \vec{v}(t_0) = \vec{v}\right] \tag{52}$$

Substituting all at once,

subs(**(52)**, **(51)**)

$$\vec{A}_t = \frac{e\,\vec{v}\,(t_{0_t})\,(\vec{v} \cdot \vec{R})}{\left(\|\vec{R}\| - \frac{\vec{R} \cdot \vec{v}}{c}\right)^2 c\,\|\vec{R}\|} - \frac{e\,\vec{v}\,(t_{0_t})\,(\vec{v} \cdot \vec{v})}{\left(\|\vec{R}\| - \frac{\vec{R} \cdot \vec{v}}{c}\right)^2 c^2} + \frac{e\,\vec{v}\,(t_{0_t})\,(\vec{R} \cdot \vec{a})}{\left(\|\vec{R}\| - \frac{\vec{R} \cdot \vec{v}}{c}\right)^2 c^2} \tag{53}$$

$$+\frac{e\,\vec{a}\left(t_{0_t}\right)}{\left(\|\vec{R}\|-\frac{\vec{R}\cdot\vec{v}}{c}\right)c}$$

simplify((**53**))

$$\vec{A}_t=\frac{e\left(t_{0_t}\right)\left(\|\vec{R}\|^2\,\vec{a}\,c-\vec{v}\,\|\vec{v}\|^2\,\|\vec{R}\|-\|\vec{R}\|\,(\vec{R}\cdot\vec{v})\,\vec{a}+\vec{v}\,(\vec{R}\cdot\vec{a})\,\|\vec{R}\|+c\,\vec{v}\,(\vec{R}\cdot\vec{v})\right)}{\left(c\,\|\vec{R}\|-\vec{R}\cdot\vec{v}\right)^2\,\|\vec{R}\|} \quad \textbf{(54)}$$

This expression involves the derivative of the retarded time $\frac{\partial t_0}{\partial t}$. So, from **(41)**

(41)

$$t_0=t-\frac{\|\vec{r}-\vec{r}_0(t_0)\|}{c} \quad \textbf{(55)}$$

diff((**41**), t)

$$t_{0_t}=1+\frac{\left(t_{0_t}\right)\left(\mathrm{D}(\vec{r}_0)(t_0)\cdot(\vec{r}-\vec{r}_0(t_0))\right)}{\|\vec{r}-\vec{r}_0(t_0)\|\,c} \quad \textbf{(56)}$$

Doing as in **(53)**,
subs((**52**), (**56**))

$$t_{0_t}=1+\frac{\left(t_{0_t}\right)(\vec{v}\cdot\vec{R})}{\|\vec{R}\|\,c} \quad \textbf{(57)}$$

isolate((**57**), *lhs*((**57**)))

$$t_{0_t}=\frac{1}{1-\frac{\vec{R}\cdot\vec{v}}{\|\vec{R}\|\,c}} \quad \textbf{(58)}$$

In this way we get for $\frac{\partial}{\partial t}\vec{A}$

subs((**58**), (**54**))

$$\vec{A}_t=\frac{e\left(\|\vec{R}\|^2\,\vec{a}\,c-\vec{v}\,\|\vec{v}\|^2\,\|\vec{R}\|-\|\vec{R}\|\,(\vec{R}\cdot\vec{v})\,\vec{a}+\vec{v}\,(\vec{R}\cdot\vec{a})\,\|\vec{R}\|+c\,\vec{v}\,(\vec{R}\cdot\vec{v})\right)}{\left(1-\frac{\vec{R}\cdot\vec{v}}{\|\vec{R}\|\,c}\right)\left(c\,\|\vec{R}\|-\vec{R}\cdot\vec{v}\right)^2\,\|\vec{R}\|} \quad \textbf{(59)}$$

Collecting the results of the two previous subsections, we have for the electric field
$\vec{E}(X)=-\frac{1}{c}\left(\frac{\partial}{\partial t}\vec{A}(X)\right)-\nabla\varphi(X)$

$$\vec{E}=-\frac{\vec{A}_t}{c}-\nabla\varphi \quad \textbf{(60)}$$

subs((**49**), (**59**), (**60**))

$$\vec{E} = -\frac{e\left(\|\vec{R}\|^2 \vec{a}\, c - \vec{v}\, \|\vec{v}\|^2 \|\vec{R}\| - \|\vec{R}\| \left(\vec{R}\cdot\vec{v}\right) \vec{a} + \vec{v}\left(\vec{R}\cdot\vec{a}\right)\|\vec{R}\| + c\,\vec{v}\left(\vec{R}\cdot\vec{v}\right)\right)}{c\left(1 - \frac{\vec{R}\cdot\vec{v}}{\|\vec{R}\|\, c}\right)\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^2 \|\vec{R}\|} + \frac{c\, e\left(-\|\vec{v}\|^2 \vec{R} - \|\vec{R}\|\, c\,\vec{v} + \vec{R}\, c^2 + \left(\vec{R}\cdot\vec{a}\right)\vec{R} + \left(\vec{R}\cdot\vec{v}\right)\vec{v}\right)}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} \quad \textbf{(61)}$$

The book, presents this result as equation (63.8):

$$\vec{E} = \frac{e\left(1 - \frac{v^2}{c^2}\right)}{\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)^3}\left(\vec{R} - \frac{\vec{v}}{c}R\right) + \frac{e}{c^2\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)^3}\vec{R} \times \left(\left(\vec{R} - \frac{\vec{v}}{c}R\right) \times \vec{a}\right)$$

where $R \equiv \|\vec{R}\|$ and $v \equiv \|\vec{v}\|$. To rewrite **(61)** as in the above, introduce the two triple vector products

$R\_ \times (v\_ \times a\_)$:

$expand(\%) = \%$

$$\vec{v}\left(\vec{R}\cdot\vec{a}\right) - \left(\vec{R}\cdot\vec{v}\right)\vec{a} = \vec{R} \times \left(\vec{v} \times \vec{a}\right) \quad \textbf{(62)}$$

$simplify(\textbf{(61)}, \{\textbf{(62)}\})$

$$\vec{E} = \frac{1}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3}\left(e\left(-\|\vec{R}\|\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right) + \vec{R}\, c\left(\vec{R}\cdot\vec{a}\right) - \|\vec{R}\|^2 \vec{a}\, c + \left(-c^2\vec{v} + \vec{v}\,\|\vec{v}\|^2\right)\|\vec{R}\| + \vec{R}\, c^3 - \vec{R}\, c\,\|\vec{v}\|^2\right)\right) \quad \textbf{(63)}$$

$R\_ \times (R\_ \times a\_)$:

$expand(\%) = \%$

$$\left(\vec{R}\cdot\vec{a}\right)\vec{R} - \|\vec{R}\|^2 \vec{a} = \vec{R} \times \left(\vec{R} \times \vec{a}\right) \quad \textbf{(64)}$$

$simplify(\textbf{(63)}, \{\textbf{(64)}\})$

$$\vec{E} = \frac{\left(c\left(\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right) - \|\vec{R}\|\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right) + \left(c - \|\vec{v}\|\right)\left(c + \|\vec{v}\|\right)\left(\vec{R}\, c - \|\vec{R}\|\,\vec{v}\right)\right) e}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} \quad \textbf{(65)}$$

Split now this result into two terms, one of them involving the acceleration $\vec{a}$. For that purpose first expand the expression *without expanding the cross products*

$lhs(\textbf{(65)}) = frontend(expand, [rhs(\textbf{(65)})])$

$$\vec{E} = \frac{e\,\|\vec{R}\|\,\|\vec{v}\|^2\vec{v}}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} - \frac{e\,\|\vec{R}\|\, c^2\vec{v}}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} - \frac{e\,\|\vec{v}\|^2\vec{R}\, c}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} + \frac{e\,\vec{R}\, c^3}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} + \frac{e\, c\left(\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right)}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} - \frac{e\,\|\vec{R}\|\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right)}{\left(c\,\|\vec{R}\| - \vec{R}\cdot\vec{v}\right)^3} \quad \textbf{(66)}$$

Introduce the notation used in the textbook, $R \equiv \|\vec{R}\|$ and $v \equiv \|\vec{v}\|$ and proceed with the splitting

$lhs(\textbf{(65)}) = subs(Norm(R\_) = R, Norm(v\_) = v, add(normal([selectremove(\textbf{not}\ has, rhs(\textbf{(66)}), \vec{a})])))$

$$\vec{E} = \frac{e\left(-c^2\vec{v}\, R + \vec{v}\, v^2 R + \vec{R}\, c^3 - \vec{R}\, c\, v^2\right)}{\left(c\, R - \vec{R}\cdot\vec{v}\right)^3} - \frac{e\left(R\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right) - c\left(\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right)\right)}{\left(c\, R - \vec{R}\cdot\vec{v}\right)^3} \quad \textbf{(67)}$$

Rearrange *only the first term* using *simplify*; that can be done in different ways, perhaps the simplest is using subsop

$subsop([2,1] = simplify(op([2,1], \mathbf{(67)})), \mathbf{(67)})$

$$\vec{E} = \frac{e\,(c-v)\,(c+v)\,(-R\,\vec{v} + \vec{R}\,c)}{(c\,R - \vec{R}\cdot\vec{v})^3} - \frac{e\,(R\,(\vec{R}\times(\vec{v}\times\vec{a})) - c\,(\vec{R}\times(\vec{R}\times\vec{a})))}{(c\,R - \vec{R}\cdot\vec{v})^3} \quad \mathbf{(68)}$$

By eye this result is mathematically equal to equation **(63.8)** of the textbook, shown here above before **(62)** .

Algebraic manipulation rewriting (68) *as the textbook equation* (*63.8*)

Rewriting **(68)** closer to the form shown in the textbook as (63.8), while not a mathematical problem, it is a sort of algebraic manipulation challenge.

For the first term of **(68)**, two steps using the command subsop (useful for more surgical manipulations) suffice:

1. The numerator of the first term is homogeneous in $\vec{R}$ with degree 1 while the denominator is of degree 3, so dividing by c both $\vec{R}$ and its norm *only in the first term*, then dividing the term by c^2,

$$subsop\left([2,1] = \frac{subs\left(\left[R = \frac{R}{c}, \underline{R} = \frac{\underline{R}}{c}\right], op([2,1], \mathbf{(68)})\right)}{c^2}, \mathbf{(68)}\right)$$

$$\vec{E} = \frac{e\,(c-v)\,(c+v)\left(-\frac{R\,\vec{v}}{c} + \vec{R}\right)}{\left(R - \left(\frac{\vec{R}}{c}\right)\cdot\vec{v}\right)^3 c^2} - \frac{e\,(R\,(\vec{R}\times(\vec{v}\times\vec{a})) - c\,(\vec{R}\times(\vec{R}\times\vec{a})))}{(c\,R - \vec{R}\cdot\vec{v})^3} \quad \mathbf{(69)}$$

2. Combining now

$\frac{(c-v)\,(c+v)}{c^2} : \% = expand(\%)$

$$\frac{(c-v)\,(c+v)}{c^2} = 1 - \frac{v^2}{c^2} \quad \mathbf{(70)}$$

$$subsop\left([2,1] = op([2,1], \mathbf{(69)})\cdot\frac{rhs(\mathbf{(70)})}{lhs(\mathbf{(70)})}, \mathbf{(69)}\right)$$

$$\vec{E} = \frac{e\left(-\frac{R\,\vec{v}}{c} + \vec{R}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(R - \frac{\vec{R}\cdot\vec{v}}{c}\right)^3} - \frac{e\,(R\,(\vec{R}\times(\vec{v}\times\vec{a})) - c\,(\vec{R}\times(\vec{R}\times\vec{a})))}{(c\,R - \vec{R}\cdot\vec{v})^3} \quad \mathbf{(71)}$$

The manipulation for the second term of **(71)** is analogous, using subsop and subsindets

1. The numerator of the second term is homogeneous in $\vec{R}$ of degree 2 while the denominator is of degree 3, so dividing $\vec{R}$ and its norm by c, then the whole term by c we get

$$eval\left(subsop\left([2,2]=\frac{subs\left(\left[R=\frac{R}{c}, R\_=\frac{R\_}{c}\right], op([2,2], \mathbf{(71)})\right)}{c}, \mathbf{(71)}\right)\right)$$

$$\vec{E}=\frac{e\left(-\frac{R\,\vec{v}}{c}+\vec{R}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(R-\frac{\vec{R}\cdot\vec{v}}{c}\right)^3}-\frac{e\left(\frac{R\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right)}{c^2}-\frac{\vec{R}\times\left(\vec{R}\times\vec{a}\right)}{c}\right)}{\left(R-\frac{\vec{R}\cdot\vec{v}}{c}\right)^3 c} \quad \mathbf{(72)}$$

2. The second term is now homogeneous in *triple vector products*; one quick way of taking advantage of that is using *subsindets* with its option *flat* to multiply each triple product by *c* and dividing the whole term by *c*

$$subsop\left([2,2]=\frac{subsindets_{flat}\left(op([2,2], \mathbf{(72)}), specfunc(\grave{}\&x\grave{}), u\mapsto c\,u\right)}{c}, \mathbf{(72)}\right)$$

$$\vec{E}=\frac{e\left(-\frac{R\,\vec{v}}{c}+\vec{R}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(R-\frac{\vec{R}\cdot\vec{v}}{c}\right)^3}-\frac{e\left(\frac{R\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right)}{c}-\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right)}{\left(R-\frac{\vec{R}\cdot\vec{v}}{c}\right)^3 c^2} \quad \mathbf{(73)}$$

The same operation with a further tweak arrives closer to the form shown in the book, i.e. instead of just the above, also divide and multiply by *c* only the triple vector product that involves $\vec{v}$

$$subsop\left([2,2]=\frac{1}{c}\left(subsindets_{flat}\left(op([2,2], \mathbf{(72)}), specfunc(\grave{}\&x\grave{}), u\mapsto \mathbf{if}\ has(u,\vec{v})\ \mathbf{then}\ c^2\ subs\left(\vec{v}=\vec{v}\,\frac{1}{c}, u\right)\ \mathbf{else}\ c\,u\ \mathbf{end\ if}\right)\right), \mathbf{(72)}\right)$$

$$\vec{E}=\frac{e\left(-\frac{R\,\vec{v}}{c}+\vec{R}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(R-\frac{\vec{R}\cdot\vec{v}}{c}\right)^3}-\frac{e\left(R\left(\vec{R}\times\left(\left(\frac{\vec{v}}{c}\right)\times\vec{a}\right)\right)-\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right)}{\left(R-\frac{\vec{R}\cdot\vec{v}}{c}\right)^3 c^2} \quad \mathbf{(74)}$$

The magnetic field $\vec{H}=\nabla\times\vec{A}$

The book does not show an explicit form for $\vec{H}$, it only indicates that it is related to the electric field by the formula

$$\vec{H}=\frac{1}{\|\vec{R}\|}\left(\vec{R}\times\vec{E}\right)$$

Thus in this section we compute the explicit form of $\vec{H}$ and show that this relationship mentioned in the book holds. To compute $\vec{H}=\nabla\times\vec{A}$ we proceed as done in the previous sections, the right-hand side should be taken at the previous (retarded) time t_0. For clarity, turn OFF the compact display of functions.

OFF;

We need to calculate

$H\_(X) = Curl\left(A\_\left(x, y, z, t_0(x, y, z, t)\right)\right)$

$$\vec{H}(X) = \nabla \times \vec{A}\left(x, y, z, t_0(X)\right) \tag{75}$$

The chain rule $\nabla \times \vec{A}\left(t_0(x, y, z, t)\right) = \nabla \times \vec{A} + \left(\nabla t_0(x, y, z, t)\right) \times \left(\frac{\partial}{\partial t_0} \vec{A}(t_0)\right)$

To compute the curl in **(75)** we need to take into account the chain rule with regards to dependency through $t_0(x, y, z, t)$. Although using tensor notation this rule is straightforward, it is interesting to see how hoe could derive it using 3D vector notation. One way of doing that is to expand the vector potential $\vec{A}$ in components and let the system automatically perform the chain rule, then go back to the abstract (non-component) vector notation.

So introduce

$A\_\left(x, y, z, t_0(x, y, z, t)\right) = A_x\left(x, y, z, t_0(x, y, z, t)\right) \cdot \_i + A_y\left(x, y, z, t_0(x, y, z, t)\right) \cdot \_j + A_z\left(x, y, z, t_0(x, y, z, t)\right) \cdot \_k$

$$\vec{A}\left(x, y, z, t_0(X)\right) = A_x\left(x, y, z, t_0(X)\right) \hat{i} + A_y\left(x, y, z, t_0(X)\right) \hat{j} + A_z\left(x, y, z, t_0(X)\right) \hat{k} \tag{76}$$

$eval(\textbf{(75)}, \textbf{(76)})$

$$\vec{H}(X) = \left(D_2(A_z)\left(x, y, z, t_0(X)\right) + D_4(A_z)\left(x, y, z, t_0(X)\right)\left(\frac{\partial}{\partial y} t_0(X)\right) - D_3(A_y)\left(x, y, z, t_0(X)\right) - D_4(A_y)\left(x, y, z, t_0(X)\right)\left(\frac{\partial}{\partial z} t_0(X)\right)\right)\hat{i} + \left(D_3(A_x)\left(x, y, z, t_0(X)\right) + D_4(A_x)\left(x, y, z, t_0(X)\right)\left(\frac{\partial}{\partial z} t_0(X)\right) - D_1(A_z)\left(x, y, z, t_0(X)\right) - D_4(A_z)\left(x, y, z, t_0(X)\right)\left(\frac{\partial}{\partial x} t_0(X)\right)\right)\hat{j} + \left(D_1(A_y)\left(x, y, z, t_0(X)\right) + D_4(A_y)\left(x, y, z, t_0(X)\right)\left(\frac{\partial}{\partial x} t_0(X)\right) - D_2(A_x)\left(x, y, z, t_0(X)\right) - D_4(A_x)\left(x, y, z, t_0(X)\right)\left(\frac{\partial}{\partial y} t_0(X)\right)\right)\hat{k} \tag{77}$$

We know, by construction, that this is the result of applying the chain rule so one of the terms in this expression must be the curl of $\vec{A}$ at fixed t_0. To see that, remove the functionality of the retarded time and express the result using *diff* instead of D

$convert\left(subs\left(\left(x, y, z, t_0(x, y, z, t)\right) = \left(x, y, z, t_0\right), \textbf{(77)}\right), diff\right)$

$$\vec{H}(X) = \left(\frac{\partial}{\partial y} A_z(x, y, z, t_0) + \left(\frac{\partial}{\partial t_0} A_z(x, y, z, t_0)\right)\left(\frac{\partial}{\partial y} t_0(X)\right) - \frac{\partial}{\partial z} A_y(x, y, z, t_0) - \left(\frac{\partial}{\partial t_0} A_y(x, y, z, t_0)\right)\left(\frac{\partial}{\partial z} t_0(X)\right)\right)\hat{i} + \left(\frac{\partial}{\partial z} A_x(x, y, z, t_0) + \left(\frac{\partial}{\partial t_0} A_x(x, y, z, t_0)\right)\left(\frac{\partial}{\partial z} t_0(X)\right) - \frac{\partial}{\partial x} A_z(x, y, z, t_0) - \left(\frac{\partial}{\partial t_0} A_z(x, y, z, t_0)\right)\left(\frac{\partial}{\partial x} t_0(X)\right)\right)\hat{j} + \left(\frac{\partial}{\partial x} A_y(x, y, z, t_0) + \left(\frac{\partial}{\partial t_0} A_y(x, y, z, t_0)\right)\left(\frac{\partial}{\partial x} t_0(X)\right) - \frac{\partial}{\partial y} A_x(x, y, z, t_0) - \left(\frac{\partial}{\partial t_0} A_x(x, y,\right. \tag{78}$$

$$z, t_0)\Big)\left(\frac{\partial}{\partial y} t_0(X)\right)\Big)\hat{k}$$

Introduce the non-expanded form of the curl at fixed t_0

$subs\left(t_0(x, y, z, t) = t_0, \textbf{(76)}\right)$

$$\vec{A}(x, y, z, t_0) = A_x(x, y, z, t_0)\,\hat{i} + A_y(x, y, z, t_0)\,\hat{j} + A_z(x, y, z, t_0)\,\hat{k} \qquad \textbf{(79)}$$

$Curl(rhs(\textbf{(79)})) = \%Curl\left(A\_(x, y, z, t_0)\right)$

$$\left(\frac{\partial}{\partial y} A_z(x, y, z, t_0) - \frac{\partial}{\partial z} A_y(x, y, z, t_0)\right)\hat{i} + \left(\frac{\partial}{\partial z} A_x(x, y, z, t_0) - \frac{\partial}{\partial x} A_z(x, y, z, t_0)\right)\hat{j} + \left(\frac{\partial}{\partial x} A_y(x, y, z, t_0) - \frac{\partial}{\partial y} A_x(x, y, z, t_0)\right)\hat{k} = \nabla\times\vec{A}(x, y, z, t_0) \qquad \textbf{(80)}$$

$collect(simplify(\textbf{(78)}, \{\textbf{(80)}\}), [\_i, \_j, \_k])$

$$\vec{H}(X) = \hat{i}\left(\left(\frac{\partial}{\partial t_0} A_z(x, y, z, t_0)\right)\left(\frac{\partial}{\partial y} t_0(X)\right) - \left(\frac{\partial}{\partial t_0} A_y(x, y, z, t_0)\right)\left(\frac{\partial}{\partial z} t_0(X)\right)\right) + \hat{j}\left(-\left(\frac{\partial}{\partial t_0} A_z(x, y, z, t_0)\right)\left(\frac{\partial}{\partial x} t_0(X)\right) + \left(\frac{\partial}{\partial t_0} A_x(x, y, z, t_0)\right)\left(\frac{\partial}{\partial z} t_0(X)\right)\right) + \hat{k}\left(-\left(\frac{\partial}{\partial t_0} A_x(x, y, z, t_0)\right)\left(\frac{\partial}{\partial y} t_0(X)\right) + \left(\frac{\partial}{\partial t_0} A_y(x, y, z, t_0)\right)\left(\frac{\partial}{\partial x} t_0(X)\right)\right) + \nabla\times\vec{A}(x, y, z, t_0) \qquad \textbf{(81)}$$

Indeed, the last term is the curl at fixed t_0. To understand the first part of this expression $\left(\hat{i}\ (\ldots) + \hat{j}\ (\ldots) + \hat{k}\ (\ldots)\right)$, take a look at the $\hat{i}$ component

$op(1, rhs(\textbf{(81)}))$

$$\hat{i}\left(\left(\frac{\partial}{\partial t_0} A_z(x, y, z, t_0)\right)\left(\frac{\partial}{\partial y} t_0(X)\right) - \left(\frac{\partial}{\partial t_0} A_y(x, y, z, t_0)\right)\left(\frac{\partial}{\partial z} t_0(X)\right)\right) \qquad \textbf{(82)}$$

This is the $\hat{i}$ term of $\left(\nabla t_0\right) \times \dfrac{\partial \vec{A}}{\partial\, t_0}$

$\%Gradient\left(t_0(x, y, z, t)\right) \times diff\left(A\_(x, y, z, t_0), t_0\right)$

$$\left(\nabla t_0(X)\right) \times \left(\frac{\partial}{\partial t_0} \vec{A}(x, y, z, t_0)\right) \qquad \textbf{(83)}$$

$A\_(x, y, z, t_0) = A_x(x, y, z, t_0)\cdot\_i + A_y(x, y, z, t_0)\cdot\_j + A_z(x, y, z, t_0)\cdot\_k$

$$\vec{A}(x, y, z, t_0) = A_x(x, y, z, t_0)\,\hat{i} + A_y(x, y, z, t_0)\,\hat{j} + A_z(x, y, z, t_0)\,\hat{k} \qquad \textbf{(84)}$$

$value(subs(\textbf{(84)}, \textbf{(83)})) = \textbf{(83)}$

$$\left(\hat{i}\left(\frac{\partial}{\partial x} t_0(X)\right) + \hat{k}\left(\frac{\partial}{\partial z} t_0(X)\right) + \hat{j}\left(\frac{\partial}{\partial y} t_0(X)\right)\right) \times \left(\frac{\partial}{\partial t_0}\left(A_x(x, y, z, t_0)\,\hat{i} + A_y(x, y, z, t_0)\,\hat{j} + A_z(x, y, z, t_0)\,\hat{k}\right)\right) = \left(\nabla t_0(X)\right) \times \left(\frac{\partial}{\partial t_0} \vec{A}(x, y, z, t_0)\right) \qquad \textbf{(85)}$$

Using this, the expression **(81)** for $\vec{H}$, the curl of $\vec{A}$, becomes

$simplify((\mathbf{81}), \{(\mathbf{85})\})$

$$\vec{H}(X) = \nabla\times \vec{A}(x, y, z, t_0) + (\nabla t_0(X)) \times \left(\frac{\partial}{\partial t_0} \vec{A}(x, y, z, t_0)\right) \quad \mathbf{(86)}$$

where we see how the chain rule shall be applied.

So applying to **(75)** the chain rule derived in the previous subsection we have

$H\_(X) = \%Curl(A\_(x, y, z, t_0)) + \%Gradient(t_0(X)) \times diff(A\_(x, y, z, t_0), t_0)$

$$\vec{H}(X) = \nabla\times \vec{A}(x, y, z, t_0) + (\nabla t_0(X)) \times \left(\frac{\partial}{\partial t_0} \vec{A}(x, y, z, t_0)\right) \quad \mathbf{(87)}$$

where t_0 is taken as a function of (x, y, z, t) only in $\nabla t_0(X)$. Now that the functionality is understood, turning ON the compact display of functions and displaying the fields by their names,

$CompactDisplay((\mathbf{87}), E\_(X))$

$$\vec{A}(x, y, z, t_0) \text{ will now be displayed as } \vec{A}$$

$$\vec{H}(x, y, z, t) \text{ will now be displayed as } \vec{H}$$

$$t_0(x, y, z, t) \text{ will now be displayed as } t_0$$

$$\vec{E}(x, y, z, t) \text{ will now be displayed as } \vec{E} \quad \mathbf{(88)}$$

The value of $\nabla t_0(X)$ is computed lines above as **(48)**

(48)

$$\nabla t_0 = \frac{\vec{R}}{-c \, \|\vec{R}\| + \vec{R} \cdot \vec{v}} \quad \mathbf{(89)}$$

The expression for $\vec{A}$ with no dependency is computed lines above, as **(28)**,

$subs(A\_ = A\_(x, y, z, t_0), (\mathbf{28}))$

$$\vec{A} = \frac{e\,\vec{v}}{\left(\|\vec{R}\| - \frac{\vec{R}\cdot\vec{v}}{c}\right) c} \quad \mathbf{(90)}$$

The expressions for $\vec{R}$ and the velocity in terms of t_0 with no dependency are

$R\_ = r\_(x, y, z) - r_{0\_}(t_0), v\_ = v\_(t_0)$

$$\vec{R} = \vec{r}(x, y, z) - \vec{r}_0(t_0), \vec{v} = \vec{v}(t_0) \quad \mathbf{(91)}$$

$CompactDisplay(r\_(x, y, z))$

$$\vec{r}(x, y, z) \text{ will now be displayed as } \vec{r} \quad \mathbf{(92)}$$

$subs((\mathbf{91}), [(\mathbf{89}), (\mathbf{90})])$

$$\left[\nabla t_0 = \frac{\vec{r} - \vec{r}_0(t_0)}{-c \, \|\vec{r} - \vec{r}_0(t_0)\| + (\vec{r} - \vec{r}_0(t_0)) \cdot \vec{v}(t_0)}, \vec{A}\right. \quad \mathbf{(93)}$$

$$= \frac{e\,\vec{v}(t_0)}{\left(\left\|\vec{r}-\vec{r}_0(t_0)\right\| - \frac{\left(\vec{r}-\vec{r}_0(t_0)\right)\cdot\vec{v}(t_0)}{c}\right) c} \Bigg]$$

Introducing this into $\vec{H}(X) = \nabla \times \vec{A}\left(x, y, z, t_0\right) + \left(\nabla t_0(X)\right) \times \left(\frac{\partial \vec{A}}{\partial t_0}\right)$,

eval(**(87)**, **(93)**)

$$\vec{H} = \nabla \times \left(\frac{e\,\vec{v}(t_0)}{\left(\left\|\vec{r}-\vec{r}_0(t_0)\right\| - \frac{\left(\vec{r}-\vec{r}_0(t_0)\right)\cdot\vec{v}(t_0)}{c}\right) c}\right) \tag{94}$$

$$+ \frac{1}{-c\left\|\vec{r}-\vec{r}_0(t_0)\right\| + \left(\vec{r}-\vec{r}_0(t_0)\right)\cdot\vec{v}(t_0)} \Bigg(\left(\vec{r}-\vec{r}_0(t_0)\right) \times \Bigg($$

$$- \frac{1}{\left(\left\|\vec{r}-\vec{r}_0(t_0)\right\| - \frac{\left(\vec{r}-\vec{r}_0(t_0)\right)\cdot\vec{v}(t_0)}{c}\right)^2 c} \Bigg(e\,\vec{v}(t_0) \Bigg($$

$$- \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{r}_0(t_0)\right)\cdot\left(\vec{r}-\vec{r}_0(t_0)\right)}{\left\|\vec{r}-\vec{r}_0(t_0)\right\|}$$

$$- \frac{-\left(\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{r}_0(t_0)\right)\cdot\vec{v}(t_0) + \left(\vec{r}-\vec{r}_0(t_0)\right)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{v}(t_0)\right)}{c} \Bigg)\Bigg)$$

$$+ \frac{e\left(\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{v}(t_0)\right)}{\left(\left\|\vec{r}-\vec{r}_0(t_0)\right\| - \frac{\left(\vec{r}-\vec{r}_0(t_0)\right)\cdot\vec{v}(t_0)}{c}\right) c} \Bigg)\Bigg)$$

Before computing the first term $\nabla \times (\,\dots)$, for readability, re-introduce the velocity $\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{r}_0(t_0) = \vec{v}$, the acceleration $\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{v}(t_0) = \vec{a}$, then remove the dependency of these functions on t_0, not relevant anymore since there are no more derivatives with respect to t_0. Performing these substitutions in sequence,

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\vec{r}_0(t_0) = \vec{v},\ \frac{\mathrm{d}}{\mathrm{d}t_0}\vec{v}(t_0) = \vec{a},\ \vec{v}(t_0) = \vec{v},\ \vec{r}_0(t_0) = \vec{r}_0$$

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\,\vec{r}_0(t_0)=\vec{v},\ \frac{\mathrm{d}}{\mathrm{d}t_0}\,\vec{v}(t_0)=\vec{a},\vec{v}(t_0)=\vec{v},\vec{r}_0(t_0)=\vec{r}_0 \tag{95}$$

subs(**(95)**, **(94)**)

$$\vec{H}=\nabla\times\left(\frac{e\,\vec{v}}{\left(\left\|\vec{r}-\vec{r}_0\right\|-\frac{(\vec{r}-\vec{r}_0)\cdot\vec{v}}{c}\right)c}\right)+\frac{1}{-c\left\|\vec{r}-\vec{r}_0\right\|+(\vec{r}-\vec{r}_0)\cdot\vec{v}}\left((\vec{r}-\vec{r}_0)\times\left(-\frac{e\,\vec{v}\left(-\frac{\vec{v}\cdot(\vec{r}-\vec{r}_0)}{\left\|\vec{r}-\vec{r}_0\right\|}-\frac{-\vec{v}\cdot\vec{v}+(\vec{r}-\vec{r}_0)\cdot\vec{a}}{c}\right)}{\left(\left\|\vec{r}-\vec{r}_0\right\|-\frac{(\vec{r}-\vec{r}_0)\cdot\vec{v}}{c}\right)^2 c}+\frac{e\,\vec{a}}{\left(\left\|\vec{r}-\vec{r}_0\right\|-\frac{(\vec{r}-\vec{r}_0)\cdot\vec{v}}{c}\right)c}\right)\right) \tag{96}$$

Activate now the inert curl $\nabla\times(\ldots)$

value(**(96)**)

$$\vec{H}=\frac{1}{c}\left(e\left(\left(-\frac{c^2\,\hat{i}\left(\left(\frac{\partial}{\partial x}\,\vec{r}\right)\cdot(\vec{r}-\vec{r}_0)\right)}{\left(c\left\|\vec{r}-\vec{r}_0\right\|-(\vec{r}-\vec{r}_0)\cdot\vec{v}\right)^2\left\|\vec{r}-\vec{r}_0\right\|}+\frac{c\,\hat{i}\left(\left(\frac{\partial}{\partial x}\,\vec{r}\right)\cdot\vec{v}\right)}{\left(c\left\|\vec{r}-\vec{r}_0\right\|-(\vec{r}-\vec{r}_0)\cdot\vec{v}\right)^2}-\frac{c^2\,\hat{j}\left(\left(\frac{\partial}{\partial y}\,\vec{r}\right)\cdot(\vec{r}-\vec{r}_0)\right)}{\left(c\left\|\vec{r}-\vec{r}_0\right\|-(\vec{r}-\vec{r}_0)\cdot\vec{v}\right)^2\left\|\vec{r}-\vec{r}_0\right\|}+\frac{c\,\hat{j}\left(\left(\frac{\partial}{\partial y}\,\vec{r}\right)\cdot\vec{v}\right)}{\left(c\left\|\vec{r}-\vec{r}_0\right\|-(\vec{r}-\vec{r}_0)\cdot\vec{v}\right)^2}-\frac{c^2\,\hat{k}\left(\left(\frac{\partial}{\partial z}\,\vec{r}\right)\cdot(\vec{r}-\vec{r}_0)\right)}{\left(c\left\|\vec{r}-\vec{r}_0\right\|-(\vec{r}-\vec{r}_0)\cdot\vec{v}\right)^2\left\|\vec{r}-\vec{r}_0\right\|}+\frac{c\,\hat{k}\left(\left(\frac{\partial}{\partial z}\,\vec{r}\right)\cdot\vec{v}\right)}{\left(c\left\|\vec{r}-\vec{r}_0\right\|-(\vec{r}-\vec{r}_0)\cdot\vec{v}\right)^2}\right)\times\vec{v}\right)\right)+\frac{1}{-c\left\|\vec{r}-\vec{r}_0\right\|+(\vec{r}-\vec{r}_0)\cdot\vec{v}}\left((\vec{r}-\vec{r}_0)\times\left(-\frac{e\,\vec{v}\left(-\frac{(\vec{r}-\vec{r}_0)\cdot\vec{v}}{\left\|\vec{r}-\vec{r}_0\right\|}-\frac{-\left\|\vec{v}\right\|^2+(\vec{r}-\vec{r}_0)\cdot\vec{a}}{c}\right)}{\left(\left\|\vec{r}-\vec{r}_0\right\|-\frac{(\vec{r}-\vec{r}_0)\cdot\vec{v}}{c}\right)^2 c}\right.\right. \tag{97}$$

$$+\frac{e\,\vec{a}}{\left(\left\|\vec{r}-\vec{r}_0\right\|-\frac{\left(\vec{r}-\vec{r}_0\right)\cdot\vec{v}}{c}\right)c}\Bigg)\Bigg)$$

From **(34)** $\equiv \frac{\partial}{\partial x}\vec{r}=\hat{i},\ \frac{\partial}{\partial y}\vec{r}=\hat{j},\ \frac{\partial}{\partial z}\vec{r}=\hat{k}$, and reintroducing $\vec{r}(x,y,z)-\vec{r}_0=\vec{R}$

$subs\left(\mathbf{(34)}, \vec{r}(x,y,z)-r_{0\_}=\vec{R}, \mathbf{(97)}\right)$

$$\vec{H}=\frac{1}{c}\Bigg(e\Bigg(\Bigg(-\frac{c^2\,\hat{i}\,(\hat{i}\cdot\vec{R})}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^2\,\|\vec{R}\|}+\frac{c\,\hat{i}\,(\hat{i}\cdot\vec{v})}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^2}-\frac{c^2\,\hat{j}\,(\hat{j}\cdot\vec{R})}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^2\,\|\vec{R}\|}+\frac{c\,\hat{j}\,(\hat{j}\cdot\vec{v})}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^2}-\frac{c^2\,\hat{k}\,(\hat{k}\cdot\vec{R})}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^2\,\|\vec{R}\|}+\frac{c\,\hat{k}\,(\hat{k}\cdot\vec{v})}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^2}\Bigg)\times\vec{v}\Bigg)\Bigg)+\frac{\vec{R}\times\left(-\frac{e\,\vec{v}\left(-\frac{\vec{R}\cdot\vec{v}}{\|\vec{R}\|}-\frac{-\|\vec{v}\|^2+\vec{R}\cdot\vec{a}}{c}\right)}{\left(\|\vec{R}\|-\frac{\vec{R}\cdot\vec{v}}{c}\right)^2 c}+\frac{e\,\vec{a}}{\left(\|\vec{R}\|-\frac{\vec{R}\cdot\vec{v}}{c}\right)c}\right)}{-c\|\vec{R}\|+\vec{R}\cdot\vec{v}} \quad \mathbf{(98)}$$

$Simplify(\mathbf{(98)})$

$$\vec{H}=\frac{1}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^3\,\|\vec{R}\|}\Big(-e\,c\,(\vec{R}\times\vec{v})\,(c\|\vec{R}\|-\vec{R}\cdot\vec{v})+e\big(-c\,(\vec{R}\cdot\vec{v})+(\|\vec{v}\|^2-\vec{R}\cdot\vec{a})\,\|\vec{R}\|\big)\,(\vec{R}\times\vec{v})-e\,(c\|\vec{R}\|-\vec{R}\cdot\vec{v})\,\|\vec{R}\|\,(\vec{R}\times\vec{a})\Big) \quad \mathbf{(99)}$$

To conclude, rearrange this expression as done with the one for the electric field $\vec{E}$ at **(65)**, so first expand **(99)** *without expanding the cross products*

$lhs(\mathbf{(99)})=frontend(expand, [rhs(\mathbf{(99)})])$

$$\vec{H}=-\frac{\|\vec{R}\|\,(\vec{R}\times\vec{a})\,c\,e}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^3}+\frac{(\vec{R}\times\vec{v})\,\|\vec{v}\|^2\,e}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^3}-\frac{(\vec{R}\times\vec{v})\,c^2\,e}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^3}+\frac{(\vec{R}\cdot\vec{v})\,(\vec{R}\times\vec{a})\,e}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^3}-\frac{(\vec{R}\times\vec{v})\,(\vec{R}\cdot\vec{a})\,e}{(c\|\vec{R}\|-\vec{R}\cdot\vec{v})^3} \quad \mathbf{(100)}$$

Then introduce the notation used in the textbook, $R\equiv\|\vec{R}\|$ and $v\equiv\|\vec{v}\|$ and split into two terms, one of which contains the acceleration $\vec{a}$

$lhs(\mathbf{(100)})=subs\big(Norm(R\_)=R, Norm(v\_)=v, add(normal([selectremove(\mathbf{not}\ has, rhs(\mathbf{(100)}), \vec{a})]))\big)$

$$\vec{H}=\frac{(\vec{R}\times\vec{v})\,e\,(-c^2+v^2)}{(c\,R-\vec{R}\cdot\vec{v})^3}-\frac{e\,\big((\vec{R}\times\vec{a})\,R\,c-(\vec{R}\times\vec{a})\,(\vec{R}\cdot\vec{v})+(\vec{R}\cdot\vec{a})\,(\vec{R}\times\vec{v})\big)}{(c\,R-\vec{R}\cdot\vec{v})^3} \quad \mathbf{(101)}$$

Verifying $\vec{H}=\frac{1}{R}\left(\vec{R}\times\vec{E}\right)$

Finally, to verify that

$$\vec{H} = \frac{1}{R}\left(\vec{R}\times\vec{E}\right),$$

from the expression **(68)** for $\vec{E}$

$\frac{R\_}{R} \times$ **(68)**

$$\frac{\vec{R}\times\vec{E}}{R} = \frac{1}{R}\left(\vec{R}\times\left(\frac{e\,(c-v)\,(c+v)\,\left(-R\,\vec{v}+\vec{R}\,c\right)}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} - \frac{e\,\left(R\,\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right)-c\,\left(\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right)\right)}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}}\right)\right) \quad \textbf{(102)}$$

(101) − **(102)**

$$\vec{H} - \frac{\vec{R}\times\vec{E}}{R} = -\frac{\left(\vec{R}\times\vec{v}\right)e\,c^{2}}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} + \frac{\left(\vec{R}\times\vec{v}\right)e\,v^{2}}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} - \frac{e\,\left(\vec{R}\times\vec{a}\right)R\,c}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} + \frac{e\,\left(\vec{R}\times\vec{a}\right)\left(\vec{R}\cdot\vec{v}\right)}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} - \frac{e\,\left(\vec{R}\cdot\vec{a}\right)\left(\vec{R}\times\vec{v}\right)}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} - \frac{1}{R}\left(\vec{R}\times\left(-\frac{e\,R\,c^{2}\,\vec{v}}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} + \frac{e\,R\,v^{2}\,\vec{v}}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} + \frac{e\,\vec{R}\,c^{3}}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} - \frac{e\,\vec{R}\,c\,v^{2}}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} - \frac{e\,R\,\left(\vec{R}\times\left(\vec{v}\times\vec{a}\right)\right)}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} + \frac{e\,c\,\left(\vec{R}\times\left(\vec{R}\times\vec{a}\right)\right)}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}}\right)\right) \quad \textbf{(103)}$$

expand((**103**))

$$\vec{H} - \frac{\vec{R}\times\vec{E}}{R} = -\frac{e\,\left(\vec{R}\times\vec{a}\right)R\,c}{\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} + \frac{e\,c\,\left\|\vec{R}\right\|^{2}\left(\vec{R}\times\vec{a}\right)}{R\,\left(c\,R-\vec{R}\cdot\vec{v}\right)^{3}} \quad \textbf{(104)}$$

subs(*Norm*(*R_*) = *R*, **(104)**)

$$\vec{H} - \frac{\vec{R}\times\vec{E}}{R} = 0 \quad \textbf{(105)}$$

References

[1] Landau, L.D., and Lifshitz, E.M. **Course of Theoretical Physics Vol 2, The Classical Theory of Fields**. Elsevier, 1975.