Transformation of the Abel inverse Riccati(AIR) equation via trigonometric function

SHAOXUAN HUANG

Beijing Normal University-HongKong Baptist university united college, Jintong Road 2000, Zhuhai, Canton, China

Abstract. The integrability of the Abel differential equation is an unsolved problem since it was mentioned by the great Norway mathematician Niels Henrik Abel. Although there is no general method to tell whether a random given Abel equation is integrable or not, several solvable classes have been mentioned over the past 30 years. In this work, we consider a special non-Liouville integrable Abel equation class which is called Abel inverse Riccati(AIR) equation. Using a result mentioned by Liouville, we rewrite the AIR equation in trigonometric form. And we also obtain some similar trigonometric form of this class of Abel equation. Then, using a technique to transform the trigonometry functions into algebraic functions, we obtain the following results: 1,we conclude that a more general form of algebraic Abel equation has non-Liouville integral. 2, we obtain some new rational differential equations which have non-Liouville integral. We find methods to solve new types of Abel equation and rational differential equations. All our computation is carried out by the computer algebra system(CAS) Maple 2021/22.

Keywords: Abel differential equation · rational differential equation · Liouville integrability · Non-elementary solution

1 Introduction

Definition 1. Suppose $f_1(x) \neq 0$, The general Abel differential equation of the first kind has the form:

$$\frac{dy}{dx} = f_1(x)y^3 + f_2(x)y^2 + f_3(x)y + f_4(x) \tag{1}$$

Definition 2. Suppose $g_1(x) \neq 0$. The Abel differential equation of the second kind has the form:

$$(g_1(x)y + g_2(x))\frac{dy}{dx} = f_1(x)y^3 + f_2(x)y^2 + f_3(x)y + f_4(x) \tag{2}$$

It is shown that [7]

Theorem 1. By change of variables:

$$\frac{g_1(x)}{u} = g_1(x)y + g_2(x) \tag{3}$$

That the abel equation of the second kind is reduced to the first kind. So the Abel equation of the second kind is actually equivalent to the first kind.

Definition 3. For the first order ordinary differential equation. It is said to be **Liouville integrable**, if it can be transformed to equation that can be solved(to get rid of the differentiate operator) by integrate on both sides. The basic Liouville integrable equation including Bernoulli equation, linear equation, separable equation etc[7]

Over the past 30 years, due to the invention and evolution of the symbolic computation function of the maths software including Maple, Mathematica, many breakthrough has been made on the algorithm finding the analytical solution of the differential equation. There have been many works on the Liouville integrable case of the Abel equation, readers may refer to [7],[8],[4].

A more general "integrable" definition on the ordinary differential equation, is that we can express our solution via the power series. We already know that some first order ODEs like Riccati ODE, their solution cannot be expressed via the elementary function[17] or their integral. However, if we raise the order to second order LODE, we may compute solutions expressed by some special infinite power series functions near its singular points, like hypergeometric function, Bessel function etc[18]. By far, we cannot tell a given Abel equation is integrable, or even Liouville integrable or not. In this work, based on some existing literature, we will study the integrability of some Abel equation and rational ODE.

1.1 Preliminary

Definition 4. The Abel Inverse Riccati equation[3], is an Abel equation of the second kind of the form:

$$\frac{dy}{dx} = \frac{a_1 y^3 + a_2 y^2 + a_3 y + a_4}{(b_1 x^2 + b_2 x + b_3)y + c_1 x^2 + c_2 x + c_3} \tag{4}$$

by inverse variables $x \to y, y \to x$, this equation becomes a Riccati equation

$$\frac{dy}{dx} = \frac{(b_1 x + c_1)}{(a_1 x^3 + a_2 x^2 + a_3 x + a_4)} y^2 + \frac{(b_2 x + c_2)}{(a_1 x^3 + a_2 x^2 + a_3 x + a_4)} y + \frac{(b_3 x + c_3)}{(a_1 x^3 + a_2 x^2 + a_3 x + a_4)} \tag{5}$$

Theorem 2. In general, the AIR equation is not Liouville integrable(About how to determine a rational Riccati equation is Liouville integrable, see[9]). But it is integrable.

proof. The main idea for solving the AIR equation is to divide it into classes. Rewrite the AIR equation into the form:

$$\frac{dy}{dx} = \frac{(y-\rho_1)(y-\rho_2)(y-\rho_3)}{(b_1 x^2 + b_2 x + b_3)y + c_1 x^2 + c_2 x + c_3} \tag{6}$$

class 1 If there is three distincts roots ρ_i, then, according to [5] by applying certain Mobius transformation[16]:

$$y = \frac{px + q}{rx + t} \tag{7}$$

The AIR equation is transformed to the form:

$$\frac{dy}{dx} = \frac{y(y-1)}{(b_1 x^2 + b_2 x + b_3)y + (c_1 x^2 + c_2 x + c_3)} \tag{8}$$

for some new constants b_i, c_i, Applying change of variables:

$$x \to \frac{x(1-x)\frac{dy}{dx}}{(b_1 x + c_1)y}, y \to x \tag{9}$$

class 1 is reduced to the second order Heun differential equation of the form:

$$\frac{d^2y}{dx^2} = \frac{b_1(b_2-1)x^2 + ((b_2-1)c_1 + b_1c_2)x + c_1(1+c_2)}{x(b_1x+c_1)(x-1)}\frac{dy}{dx} - \frac{(b_3x+c_3)(b_1x+c_1)}{x^2(x-1)^2}y \tag{10}$$

which has the non-Liouville solution of HeunG function(see appendix).

class 2 If there is two distincts roots ρ_i, then by applying certain Mobius transformation, the AIR equation is transformed to the form:

$$\frac{dy}{dx} = \frac{y}{(b_1x^2 + b_2x + b_3)y + (c_1x^2 + c_2x + c_3)} \tag{11}$$

for some new constants b_i, c_i. Inversing the variables x and y, and then transform the Riccati equation to the linear second order equation by change of variables[7]:

$$y \to -\frac{xy}{\frac{dy}{dx}(b_1x + c_1)} \tag{12}$$

we arrive at:

$$\frac{d^2y}{dx^2} = \frac{b_1b_2x^2 + (b_1c_2 + b_2c_1)x - b_1 + b_1b_2}{b_1x + c_1}\frac{dy}{dx} - \frac{(b_1b_3x^2 + (b_1c_3 + b_3c_1)x + c_1c_3)}{x^2}y \tag{13}$$

If we rewrite this equation into normal form, it becomes the confluent Heun equation[2] of the form:

$$\frac{d^2y}{dx^2} + (A + \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x^2} + \frac{E}{(x-1)^2})y = 0 \tag{14}$$

which has the non-Liouville solution:

$$\begin{aligned} &C_1 e^{\sqrt{-A}x} x^{\frac{\sqrt{-4D+1}}{2}+\frac{1}{2}} (x-1)^{\frac{1-4E}{2}+\frac{1}{2}} \mathrm{HeunC}(2\sqrt{-A}, \sqrt{1-4D}, \sqrt{1-4E}, B+C, -B+\frac{1}{2}, x)+ \\ &C_2 e^{\sqrt{-A}x} x^{\frac{\sqrt{-4D+1}}{2}+\frac{1}{2}} (x-1)^{\frac{1-4E}{2}+\frac{1}{2}} \mathrm{HeunC}(2\sqrt{-A}, -\sqrt{1-4D}, \sqrt{1-4E}, B+C, -B+\frac{1}{2}, x) \end{aligned} \tag{15}$$

class3 If there is one distinct roots ρ_i, then by applying certain Mobius transformation, the AIR equation is transformed to the form:

$$\frac{dy}{dx} = \frac{1}{(b_1x^2+b_2x+b_3)y+(c_1x^2+c_2x+c_3)} \tag{16}$$

Using similar transformation that we used in solving class 2, we will arrive at:

$$\frac{d^2y}{dx^2} = \frac{b_1b_2x^2+(b_1c_2+b_2c_1)x+b_1+c_1c_2}{b_1x+c_1}\frac{dy}{dx} - (b_1b_3x^2+(b_1c_3+b_3c_1)x+c_1c_3)y \tag{17}$$

Rewriting it to the normal form results in biconfluent Heun equation(see Appendix for more details) of the form[2].

$$\frac{d^2y}{dx^2} + (-x^2+Bx+C+\frac{D}{x}+\frac{E}{x^2})y = 0 \tag{18}$$

where B, C, D, E are new constants. The non-Liouville solutions of which is:

$$\begin{aligned} &C_1 x^{\frac{\sqrt{1-4E}}{2}+\frac{1}{2}} e^{\frac{x(-x+B)}{2}} \mathrm{HeunB}(\sqrt{1-4E}, B, C+\frac{B^2}{4}, 2D, -x)+ \\ &C_2 x^{\frac{\sqrt{1-4E}}{2}+\frac{1}{2}} e^{\frac{x(-x+B)}{2}} \mathrm{HeunB}(-\sqrt{1-4E}, B, C+\frac{B^2}{4}, 2D, -x) \end{aligned} \tag{19}$$

The Heun functions is the power series solutions of Heun differential equations. Although the Heun functions are not closed form functions, in [3], the author manages to transform AIR equations to the known solvable hypergeometric equation. For the three cases mentioned above, they are equivalent to Gauss hypergeometric equation, confluent hypergeometric equation and Airy equation[18].

2 Transformation process

In this section we consider the following two Abel differential equations of the second kind. The first equation:

$$(a_1y^3 + a_2y^2 + a_3y + a_4)A(x)^n = \frac{dy}{dx}(P(x)y + Q(x)) \tag{20}$$

where a_i are arbitrary constants, $n > 0$, $P(x), Q(x)$ are non-fractional algebraic expressions[1](does not contain negative power) in terms of x. $A(x)$ is two degree polynomial which has two distinct roots ρ_1, ρ_2. The second equation:

$$(a_1y^3 + a_2y^2 + a_3y + a_4) = \frac{dy}{dx}((b_1U(x) + b_2T(x) + b_3R(x))y + c_1U(x) + c_2T(x) + c_3R(x)) \tag{21}$$

where a_i, b_i, c_i are arbitrary constants, and

$$U(x) = P(x)Q(x)^{-1}A(x) \tag{22}$$

$$T(x) = A(x)^{-n+1}Q(x) \tag{23}$$

$$R(x) = A(x)^{n+1}Q(x)^{-1} \tag{24}$$

$n > 0, P(x), Q(x)$ are non-fractional algebraic expressions. $A(x)$ is two degree polynomial which has two distinct roots ρ_1, ρ_2. In the above two Abel equations, $P(x), Q(x)$ do not contain the root ρ_1, ρ_2 and a_i do not equal to zero simultaneously. Generally, it's difficult to tell whether the above two equations are Liouville integrable or not. However, by constructing counter example, we are going to prove the following theorem.

Theorem 3. $\forall a_i, A(x), n$ in (20)(21), there exists some non-Liouville integrable cases.

[1] Here we use an informal definition. We also consider function with the variables raising to a irrational number power and complex number coefficients as algebraic. We change the definition here for simplifying our discussions.

2.1 trigonometric forms of AIR equation

Lemma. It is shown by Liouville[7], that equation of the form:

$$\frac{dy}{dx} = f(x)\cos(y) + g(x)\sin(y) + h(x) \tag{25}$$

can be transformed to an arbitrary Riccati equation by the tangent half-angle substitution $y \to 2\arctan(y)$[17].

Based on this idea, we apply the inverse substitution to the equation 5, to get:

$$\frac{dy}{dx} = \frac{(b_1x + c_1)}{(a_1x^3 + a_2x^2 + a_3x + a_4)}\cos(y) + \frac{(b_2x + c_2)}{(a_1x^3 + a_2x^2 + a_3x + a_4)}\sin(y) + \frac{(b_3x + c_3)}{(a_1x^3 + a_2x^2 + a_3x + a_4)} \tag{26}$$

for some new constants a_i, b_i, c_i And inverse the variables x and y, we obtain the trigonometric forms of the AIR equation:

$$((b_1y + c_1)\cos(x) + (b_2y + c_2)\sin(x) + b_3y + c_3)\frac{dy}{dx} = a_1y^3 + a_2y^2 + a_3y + a_4 \tag{27}$$

Lemma 1. By transformation $x \to 2\arctan(x)$, (27) is reduced to (4).

Here we point out that, despite from Liouville's results, there are some other homogeneous trigonometric form of the AIR equation.

Lemma 2. By applying transformation $x \to \arcsin(x)$, the following equation is reduced to (4)

$$((b_1y + c_1)\cos(x) + (b_2y + c_2)\sec(x) + (b_3y + c_3)\tan(x))\frac{dy}{dx} = a_1y^3 + a_2y^2 + a_3y + a_4 \tag{28}$$

Lemma 3. By applying transformation $x \to \arccos(x)$, the following equation is reduced to (4)

$$((b_1y + c_1)\sin(x) + (b_2y + c_2)\csc(x) + (b_3y + c_3)\cot(x))\frac{dy}{dx} = a_1y^3 + a_2y^2 + a_3y + a_4 \tag{29}$$

The trigonometric form of the AIR equation, both (27),(28) and (29) are unsolvable in Maple.

2.2 multiple angle tangent substitution

In order to convert the trigonometric form to the algebraic form, the well known tangent half angle substitution is introduced. However, we notice that despite from the half angle substitution, the transformation:

$$x \to n\mathrm{arctan}(x), n > 0 \tag{30}$$

will also transform the equations (27),(28),(29) into algebraic form. Let us consider the exact form after such kind of substitution. For the convenience of calculation, set $n = 2n$, then, according to Euler formula and basic properties of the trigonometric functions:

$$\sin(2n\theta) = (e^{i2n\theta} - e^{-i2n\theta})/2i = \frac{((\cos(\theta) + i\sin(\theta))^{2n} - (\cos(\theta) - i\sin(\theta))^{2n})}{2i(\cos^2(\theta) + \sin^2(\theta))^n} \tag{31}$$

$$\cos(2n\theta) = (e^{i2n\theta} + e^{-i2n\theta})/2 = \frac{((\cos(\theta) + i\sin(\theta))^{2n} + (\cos(\theta) - i\sin(\theta))^{2n})}{2(\cos^2(\theta) + \sin^2(\theta))^n} \tag{32}$$

$$\tan(2n\theta) = \frac{(\cos(\theta) + i\sin(\theta))^{2n} - (\cos(\theta) - i\sin(\theta))^{2n}}{i((\cos(\theta) + i\sin(\theta))^{2n} + (\cos(\theta) - i\sin(\theta))^{2n})} \tag{33}$$

Divide $\cos^{2n}(\theta)$ on the denominator and divisor, and substitute $\theta = \arctan(x)$, we obtain:

$$\sin(2n\mathrm{arctan}(x)) = \frac{(1 + ix)^{2n} - (1 - ix)^{2n}}{2i(1 + x^2)^n} \tag{34}$$

$$\cos(2n\mathrm{arctan}(x)) = \frac{(1 + ix)^{2n} + (1 - ix)^{2n}}{2(1 + x^2)^n} \tag{35}$$

$$\tan(2n\mathrm{arctan}(x)) = \frac{(1 + ix)^{2n} - (1 - ix)^{2n}}{i((1 + ix)^{2n} + (1 - ix)^{2n})} \tag{36}$$

So the equation (27) after the substitution becomes:

$$(a_1y^3+a_2y^2+a_3y+a_4)(x^2+1)^{n-1} = \frac{dy}{dx}((b_1y+c_1)(1-ix)^{2n}+(b_2y+c_2)(1+ix)^{2n}+(b_3y+c_3)(x^2+1)^n) \tag{37}$$

for some new constants a_i, b_i, c_i. This equation is exactly of the form (20). Since it's generated from the AIR equation, it does not have Liouville integral. To transform it back to AIR, reverse the multiple angle tangent substitution, and apply the tangent half angle substitution. the equation (28) after substitution becomes:

$$(a_1y^3 + a_2y^2 + a_3y + a_4) = \frac{dy}{dx}((b_1y + c_1)T(x) + (b_2y + c_2)R(x) + (b_3y + c_3)U(x))(x^2 + 1) \quad (38)$$

for some new constants a_i, b_i, c_i, where:

$$T(x) = \frac{(1 + ix)^{2n} - (1 - ix)^{2n}}{(1 + ix)^{2n} + (1 - ix)^{2n}}, R(x) = \frac{(1 + ix)^{2n} + (1 - ix)^{2n}}{(x^2 + 1)^n}, U(x) = R(x)^{-1} \quad (39)$$

The equation (29) after substitution becomes:

$$(a_1y^3 + a_2y^2 + a_3y + a_4) = \frac{dy}{dx}((b_1y + c_1)T(x) + (b_2y + c_2)R(x) + (b_3y + c_3)U(x))(x^2 + 1) \quad (40)$$

for some new constants a_i, b_i, c_i, where:

$$T(x) = \frac{(1 + ix)^{2n} + (1 - ix)^{2n}}{(1 + ix)^{2n} - (1 - ix)^{2n}}, R(x) = \frac{(1 + ix)^{2n} - (1 - ix)^{2n}}{(x^2 + 1)^n}, U(x) = R(x)^{-1} \quad (41)$$

Both (38) and (40) belong to the form of (21). To transform it back to AIR equation, inverse the multiple angle tangent substitution, and then apply the sine/cosine transformation. Therefore, we conclude the seconds results in this paper:

Theorem 4.(37),(38) and (40) are integrable.

For the non-rational cases of (37),(38) and (40), Maple cannot give the results. Based on the (37),(38) and (40), by applying scaling transformation $x \to \alpha x$ and shift transformation $x \to x + \beta$, we will obtain a more general form. For (37) it becomes:

$$\begin{aligned} &(a_1y^3 + a_2y^2 + a_3y + a_4)[(x - \rho_1)(x - \rho_2)]^{n-1} = \\ &\frac{dy}{dx}((b_1y + c_1)(x - \rho_1)^{2n} + (b_2y + c_2)(x - \rho_2)^{2n} + (b_3x + c_3)[(x - \rho_1)(x - \rho_2)]^n) \end{aligned} \quad (42)$$

where a_i, b_i, c_i, ρ_i can be arbitrary constants. So we are able to claim for any $a_i, A(x), n$ in (16), there exists some cases which are not Liouville integrable. For (38),(40) it becomes:

$$(a_1y^3+a_2y^2+a_3y+a_4) = \frac{dy}{dx}((b_1y+c_1)T(x)+(b_2y+c_2)R(x)+(b_3y+c_3)U(x))(x-\rho_1)(x-\rho_2) \quad (43)$$

where a_i, b_i, c_i, ρ_i can be arbitrary constants, and :

$$T(x) = \frac{(x-\rho_1)^{2n}+(x-\rho_2)^{2n}}{(x-\rho_1)^{2n}-(x-\rho_2)^{2n}}, R(x) = \frac{(x-\rho_1)^{2n}-(x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n}, U(x) = R(x)^{-1} \quad (44)$$

for (38),

$$T(x) = \frac{(x-\rho_1)^{2n}-(x-\rho_2)^{2n}}{(x-\rho_1)^{2n}+(x-\rho_2)^{2n}}, R(x) = \frac{(x-\rho_1)^{2n}+(x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n}, U(x) = R(x)^{-1} \quad (45)$$

for (40). It is of the form of (20), so we can claim that for arbitrary chosen $a_i, A(x), n$ in (20) there are some non-Liouville integrable cases.We have the third main results in this paper:

Theorem 5.(42) and two different forms of (43) are integrable.

For the rational cases of (42)(43), we can find out the rational transformation from the orginal AIR equation to a new derived equation according to the invariant theory of the Abel equation[13], this is already implemented in Maple. However, for the non-rational cases, it will be difficult to find out the transformations. Here is an non-solvable example in Maple:

$$yx^{\frac{1}{3}}(x+4)^{\frac{1}{3}}(x^{\frac{8}{3}}-(x+4)^{\frac{8}{3}}) = \frac{dy}{dx}((x^{\frac{8}{3}}-(x+4)^{\frac{8}{3}})^2y+x^{\frac{8}{3}}(x+4)^{\frac{8}{3}}) \quad (46)$$

It is of the form of (43).

Remark

1. When b_1, c_1 or b_2, c_2 equal to 0 simultaneously, equation (37) become Liouville integrable since the Riccati equation it corresponds to becomes Bernoulli equation or linear equation. When b_2, c_2 equal to 0 simultaneously, equation (38),(40) become Liouville integrable.

2. The hyperbolic function $\sinh(x), \cosh(x), \tanh(x)$ have similar properties to the trigonometric functions. In fact, applying multiple angle hyperbolic tangent substitution to the hyperbolic forms of AIR equation:

$$((b_1y+c_1)\cosh(x)+(b_2y+c_2)\sinh(x)+b_3y+c_3)\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \tag{47}$$

$$((b_1y+c_1)\cosh(x)+(b_2y+c_2)\text{sech}(x)+(b_3y+c_3)\tanh(x))\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \tag{48}$$

$$((b_1y+c_1)\sinh(x)+(b_2y+c_2)\text{csch}(x)+(b_3y+c_3)\coth(x))\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \tag{49}$$

will produce similar result to the above.

3 Integrable rational equation

Definition 5. The rational first order differential equation, which also called second order autonomous polynomial system, is of the form:

$$\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)} \tag{50}$$

where $M(x,y), N(x,y)$ are polynomial of two variables x,y.

In [8][15], several authors dicussed about the necessary and sufficient condition of the Liouville integrability of a given system. In [1],[10],[12],[11] several authors consider in what conditions can a certain type of system be reduced to Abel equation or Riccati equation.In [6][14], author study the conditions for which a polynomial system can be reduced to a second order linear equation, so the solutions in terms of special functions are available. Although these authors gave the general form of such system. It's quite difficult to judge whether a system satisfies these conditions. In this work, we consider a special case of the equation (27):

$$((b_1y+c_1)\cos(x)+(b_2y+c_2)\sin(x)+b_3y+c_3)\frac{dy}{dx}=a_1y^3+a_1y+a_2y^2+a_2 \tag{51}$$

Theorem 6. By change of variables $y \to \tan(y)$, (51) is converted to the trigonometry form:

$$((b_1\tan(y)+c_1)\cos(x)+(b_2\tan(y)+c_2)\sin(x)+b_3\tan(y)+c_3)\frac{dy}{dx}=a_1\tan(y)+a_2 \tag{52}$$

Similarly we have the following properties:

Theorem 7. for (28) and (29), if $a_1 = a_3, a_2 = a_4$, by change of variables $y \to \tan(y)$, they are converted to:

$$((b_1\tan(y)+c_1)\cos(x)+(b_2\tan(y)+c_2)\sec(x)+(b_3\tan(y)+c_3)\tan(x))\frac{dy}{dx}=a_1\tan(y)+a_2 \tag{53}$$

$$((b_1\tan(y)+c_1)\sin(x)+(b_2\tan(y)+c_2)\csc(x)+(b_3\tan(y)+c_3)\cot(x))\frac{dy}{dx}=a_1\tan(y)+a_2 \tag{54}$$

respectively. These trigonometric forms are unsolvable in Maple. We got our fourth results. For the above equation, apply multiple tangent substitution to y, x:

$$y \to 2m\mathrm{arctan}(y), x \to 2n\mathrm{arctan}(x), m, n \in \mathrm{N} \tag{55}$$

we will obtain some new integrable rational differential equations. Equation (52) becomes:

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_2(x)+b_3R_1(y)+c_3)(x^2+1)\frac{dy}{dx}=(a_1R_1(y)+a_2)(y^2+1) \tag{56}$$

for some new constants a_i, b_i, c_i. Equation (53) becomes:

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_1(x)^{-1}+(b_3R_1(y)+c_3)T_3(x))(x^2+1)\frac{dy}{dx}=(a_1R_1(y)+a_2)(y^2+1) \tag{57}$$

for some new constants a_i, b_i, c_i. Equation (54) becomes:

$$((b_1R_1(y)+c_1)T_2(x)+(b_2R_1(y)+c_2)T_2(x)^{-1}+(b_3R_1(y)+c_3)T_3(x)^{-1})(x^2+1)\frac{dy}{dx}=(a_1R_1(y)+a_2)(y^2+1) \tag{58}$$

for some new constants. In the above equation:

$$T_1(x) = \frac{(1+ix)^{2n} + (1-ix)^{2n}}{(1+x^2)^n} \tag{59}$$

$$T_2(x) = \frac{(1+ix)^{2n} - (1-ix)^{2n}}{(1+x^2)^n} \tag{60}$$

$$T_3(x) = \frac{(1+ix)^{2n} - (1-ix)^{2n}}{(1+ix)^{2n} + (1-ix)^{2n}} \tag{61}$$

$$R_1(y) = \frac{(1+iy)^{2m} - (1-iy)^{2m}}{(1+iy)^{2m} + (1-iy)^{2m}} \tag{62}$$

We got our fifth results:

Theorem 8.(56),(57) and (58) are integrable.

Like mentioned before, for (52),(53) and (54), changing the trigonometric functions in terms of x or y into the corresponding hyperbolic function, it can still be transformed back to the AIR equation. We apply the multiple angle tangent or hyperbolic tangent substitution according to the function type, and apply scaling transformation and shift transformation on x and y. Consequently, we will arrive at a more general result.

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_2(x)+b_3R_1(y)+c_3)(x-\rho_1)(x-\rho_2)\frac{dy}{dx} = (a_1R_1(y)+a_2)(y-\alpha_1)(y-\alpha_2) \tag{63}$$

$$\begin{split}((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_1(x)^{-1}+(b_3R_1(y)+c_3)T_3(x))(x-\rho_1)(x-\rho_2)\frac{dy}{dx} = \\ (a_1R_1(y)+a_2)(y-\alpha_1)(y-\alpha_2)\end{split} \tag{64}$$

$$\begin{split}((b_1R_1(y)+c_1)T_2(x)+(b_2R_1(y)+c_2)T_2(x)^{-1}+(b_3R_1(y)+c_3)T_3(x)^{-1})(x-\rho_1)(x-\rho_2)\frac{dy}{dx} = \\ (a_1R_1(y)+a_2)(y-\alpha_1)(y-\alpha_2)\end{split} \tag{65}$$

where:

$$T_1(x) = \frac{(x-\rho_1)^{2n} + (x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n} \tag{66}$$

$$T_2(x) = \frac{(x-\rho_1)^{2n} - (x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n} \tag{67}$$

$$T_3(x) = \frac{(x-\rho_1)^{2n} - (x-\rho_2)^{2n}}{(x-\rho_1)^{2n} + (x-\rho_2)^{2n}} \tag{68}$$

$$R_1(y) = \frac{(y-\alpha_1)^{2m} - (y-\alpha_2)^{2m}}{(y-\alpha_1)^{2m} + (y-\alpha_2)^{2m}} \tag{69}$$

the $a_i, b_i, c_i, \rho_i, \alpha_i$ can be arbitrary constants. We got our sixth result:

Theorem 9.(63),(64) and (65) are integrable.

Both the general cases and most special cases of (58),(59) and (60) are unsolvable in maple.

Example. Consider:

$$\frac{dy}{dx} = \frac{(y+1)^3(y-1)(8x^5-8x)}{(x^2-1)^4y^2 + 32(x^3+x)^2y + (x^2-1)^4} \tag{70}$$

This equation is of the form of (65), by transformation:

$$y \to \tanh(\frac{\tanh(y)}{2}), x \to \tanh(\frac{\operatorname{arccosh}(x)}{4}) \tag{71}$$

the (70) is reduced to AIR equation:

$$\frac{dy}{dx} = \frac{(1+y)(1-y^2)}{x^2y - y + 1} \tag{72}$$

the denominator polynomial has two distinct roots, so it is class 2 AIR equation. The solution is expressed by Kummer functions[7] (the solution of confluent hypergeometric equation):

$$\frac{(-x-\sqrt{2})\mathrm{KummerM}(-\frac{\sqrt{2}}{8}+1, 1, \frac{(1-y)\sqrt{2}}{2y+2}) - (x-\sqrt{2})\mathrm{KummerM}(-\frac{\sqrt{2}}{8}, 1, \frac{(1-\mathrm{y})\sqrt{2}}{2\mathrm{y}+2})}{(4\sqrt{2}x+8)(\frac{(-x+\sqrt{2})\mathrm{KummerU}(-\frac{\sqrt{2}}{8},1,\frac{(1-\mathrm{y})\sqrt{2}}{2\mathrm{y}+2})}{4(\sqrt{2}x+2)} + \frac{\mathrm{KummerU}(-\frac{\sqrt{2}}{8}+1,1,\frac{(1-y)\sqrt{2}}{2y+2})}{32})} + C = 0 \tag{73}$$

Remark

1. We point out that not only tangent substitution will reduce the (52)(53)(54) to algebraic forms, the multiple angle sine/cosince substitution will also work. Under such transformation, the ChebyShev polynomial will appear in the derived form. Relevant research is in progress.

2. It's also important to highlight the method introduced in our work, as our final result:

   Collary. Suppose m, n, ρ_i, α_i are constants. Consider the differential equation:

$$\frac{dy}{dx} = \frac{(y-\rho_1)(y-\rho_2)}{(x-\alpha_1)(x-\alpha_2)}\mathrm{F}(\frac{(x-\alpha_1)^{2m}-(x-\alpha_2)^{2m}}{(x-\alpha_1)^m(x-\alpha_2)^m}, \frac{(x-\alpha_1)^{2m}+(x-\alpha_2)^{2m}}{(x-\alpha_1)^m(x-\alpha_2)^m}, \frac{(x-\alpha_1)^{2m}-(x-\alpha_2)^{2m}}{(x-\alpha_1)^{2m}+(x-\alpha_2)^{2m}}, \frac{(y-\rho_1)^{2n}-(y-\rho_2)^{2n}}{(y-\rho_1)^n(y-\rho_2)^n}, \frac{(y-\rho_1)^{2n}+(y-\rho_2)^{2n}}{(y-\rho_1)^n(y-\rho_2)^n}, \frac{(y-\rho_1)^{2n}-(y-\rho_2)^{2n}}{(y-\rho_1)^{2n}+(y-\rho_2)^{2n}}) \tag{74}$$

   We can always find substitution to transform it into:

$$\frac{dy}{dx} = \mathrm{G}(\sin(x), \cos(x), \tan(x), \sin(y), \cos(y), \tan(y)) \tag{75}$$

   In which F and G are different multi variate function. This may be a useful substitution for solving equation of the form (65).

4 Conclusion

In this work, based on the AIR equation, by rewriting it to trigonometric form and apply some substitution to transfer it back to algebraic form, we obtain some new integrable Abel equation and rational equation, which are unsolvable in maths software including Maple, Mathematica etc. So this can be viewed as a new contribution to the integrable theory of the differential equation as well as symbolic computation. The Abel equation and the rational differential equation has many applications in physics, finance etc. So finding the analytical solutions of them is of great meaning.

Appendix

Heun equation

The general Heun differential equation is the of the form:

$$\frac{d^2y}{dx^2}+(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{(x-a)})\frac{dy}{dx}+\frac{\alpha\beta x-q}{x(x-1)(x-a)}y=0$$
$$(\gamma+\delta+\epsilon=\alpha+\beta+1, a\neq 0,1) \tag{76}$$

It has four singular points $0, 1, a, \infty$. The particular solution near its regular singular points is expressed by the HeunG function. The partial expansion of HeunG function is:

$$\mathrm{HeunG}(a,q,\alpha,\beta,\gamma,\delta,x)=1+\frac{qx}{a\gamma}+(\frac{\alpha\beta+\frac{q(-1-q-\alpha-\beta+\delta-a(\gamma+\delta))}{a\gamma}}{2a(1+\gamma)})x^2+O(x^3) \tag{77}$$

Through confluence processes, the General Heun Equation, transforms into four other multi-parameter equations [2], so-called Confluent , Biconfluent (BHE), Doubleconfluent (DHE) and Triconfluent Heun equation.

The Confluent Heun equation has the form:

$$\frac{d^2y}{dx^2}+(\frac{\gamma}{x-1}+\frac{\delta}{x}+\frac{\epsilon}{x(x-1)})\frac{dy}{dx}+\frac{(\alpha x-q)}{x(x-1)}y=0 \tag{78}$$

it has three singular points $0, 1, \infty$. The particular solution of the CHE is called HeunC function:

$$\mathrm{HeunC}(q,\alpha,\beta,\gamma,\delta,x)=1-\frac{qx}{\gamma}+(\frac{\alpha-\frac{q(-q+\gamma+\delta-\epsilon)}{\gamma}}{2(1+\gamma)})x^2+O(x^3) \tag{79}$$

The normal form (14) is related to the CHE by transformations:

$$y\rightarrow\frac{y}{e^{\sqrt{-A}x}x^{-\frac{\sqrt{-4D+1}}{2}+\frac{1}{2}}(x-1)^{\frac{\sqrt{1-4E}}{2}+\frac{1}{2}}} \tag{80}$$

The biconfluent Heun equation is of the form:

$$\frac{d^2y}{dx^2}+(\epsilon x+\delta+\frac{\gamma}{x})\frac{dy}{dx}+\frac{(\alpha x-q)}{x}y=0 \tag{81}$$

It has two singular points $0, \infty$. The particular solution near its regular singular point 0 is HeunB function:

$$\text{HeunB}(q, \alpha, \beta, \gamma, \delta, x) = 1 + \frac{qx}{\gamma} - (\frac{\alpha + \frac{q(-q+\delta)}{\gamma}}{2(1+\gamma)})x^2 + O(x^3) \tag{82}$$

The normal form (18) is related to BHE by transformation:

$$y \to \frac{y}{e^{\frac{x(-x+B)}{2}} x^{-\frac{\sqrt{-4E+1}}{2}+\frac{1}{2}}} \tag{83}$$

Closed form solutions of AIR

We have introduced way to convert three canonical forms of AIR to Heun equations. However, Heun functions are not closed infinite power series. In order to find the closed form solutions, [3] has found ways to convert any AIR equation to hypergeometric equations. The idea was similar: divide the AIR into three canonical forms. In Maple 2022, this algorithm has been implemented.

Three distinct roots cases

$> dsolve(diff(y(x), x) = y(x) * (y(x) - 1)/(a * (x - a[1]) * (x - a[2]) * y(x) + b * (x - b[1]) * (x - b[2])))$

$> C + (y(x) - 1)^{-a*(a[1]-a[2])+1} * (-b * (a[2] - b[1]) * (y(x) - 1) * (x - b[2]) *$

HeunC(0, -1 + a*(a[1] - a[2]), -b*(-b[2] + b[1]), 0, 2*(1/2 + (((a[2] - b[1]/2 - b[2]/2)*a[1] + (-b[1]/2 - b[2]/2)*a[2] + b[1]*b[2])*b - a[1]/2 + a[2]/2)*a)*(a[1] - a[2])/(2*a[1] - 2*a[2]), -1/(y(x) - 1)) + HeunCPrime(0, -1 + a*(a[1] - a[2]), -b*(-b[2] + b[1]), 0, 2*(1/2 + (((a[2] - b[1]/2 - b[2]/2)*a[1] + (-b[1]/2 - b[2]/2)*a[2] + b[1]*b[2])*b - a[1]/2 + a[2]/2)*a)*(a[1] - a[2])/(2*a[1] - 2*a[2]), -1/(y(x) - 1))*y(x)*(x - a[2]))/((y(x) - 1)*((x - a[2])*(a*a[1] - a*a[2] - 1)*y(x) - b*(a[2] - b[1])*(x - b[2]))*HeunC(0, 1 + (-a[1] + a[2])*a, -b*(-b[2] + b[1]), 0, 2*(1/2 + (((a[2] - b[1]/2 - b[2]/2)*a[1] + (-b[1]/2 - b[2]/2)*a[2] + b[1]*b[2])*b - a[1]/2 + a[2]/2)*a)*(a[1] - a[2])/(2*a[1] - 2*a[2]), -1/(y(x) - 1)) + HeunCPrime(0, 1 + (-a[1] + a[2])*a, -b*(-b[2] + b[1]), 0, 2*(1/2 + (((a[2] - b[1]/2 - b[2]/2)*a[1] + (-b[1]/2 - b[2]/2)*a[2] + b[1]*b[2])*b - a[1]/2 + a[2]/2)*a)*(a[1] - a[2])/(2*a[1] - 2*a[2]), -1/(y(x) - 1))*y(x)*(x - a[2])) = 0

In this special case, HeunC can be converted to closed form hypergeometric function:

$$> convert(\text{HeunC}(0, a, b, 0, c, x), \text{hypergeom})$$

$$> (1/(1-x))^{a/2+b/2+1/2+sqrt(a^2+b^2-4*c+1)/2} * \text{hypergeom}([a/2+b/2+1/2+sqrt(a^2+b^2-4*c+1)/2,$$
$$\text{a/2 - b/2 + 1/2 + sqrt}(\text{a}^2+b^2-4*c+1)/2], [a+1], x/(x-1))$$

Two distinct roots cases:

$$> dsolve(diff(y(x), x) = y(x)/(d*(x-b[1])*(x-b[2])*y(x)+c*(x-a[1])*(x-a[2])))$$

$> C + ((a[1]-b[1])*(x-b[2])*\text{KummerM}((-c*(a[2]-b[2])*(a[1]-b[1])+b[1]-b[2])/(-b[2]+b[1]),$
1 + (-a[1] + a[2])*c, y(x)*d*(-b[2] + b[1])) - KummerM(-c*(a[2] - b[2])*(a[1] - b[1]) /(-b[2] + b[1]),
1 + (-a[1] + a[2])*c, y(x)*d*(-b[2] + b[1]))*(a[1] - b[2])*(x - b[1]))/(c*(a[2] - b[1])*(a[1] - b[1])*(x -
b[2])*KummerU((-c*(a[2] - b[2])*(a[1] - b[1]) + b[1] - b[2])/(-b[2] + b[1]), 1 + (-a[1] + a[2])*c,
y(x)*d*(-b[2] + b[1])) + KummerU(-c*(a[2] - b[2])*(a[1] - b[1])/(-b[2] + b[1]), 1 + (-a[1] + a[2])*c,
y(x)*d*(-b[2] + b[1]))*(-b[2] + b[1])*(x - b[1])) = 0

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