Equivalence class of solvable Abel and Riccati equation

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1 Review

The rational differential equation, of the form:

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

where $M(x,y), N(x,y) \in \mathbb{R}[x,y]$ remains an important research interest in the study of dynamical system, physics etc. As far as we know, all integrable cases, which means we can find first integral in terms of closed form functions, are equivalent cases of the rational Riccati equation and Abel equation.
A Riccati equation is a nonlinear first order equation of the form:

$$y' = P(x)y^2 + Q(x)y + R(x), P(x) \neq 0$$

Riccati equation is converted to second order linear equation by transformation:

$$y \to -\frac{y'}{P(x)y}$$

And generally speaking, second order linear equation is easier to solve then the Riccati equation.
Abel equation is a generalization of the Riccati equation. It has two equivalent forms:

$$y' = f_1(x)y^3 + f_2(x)y^2 + f_3(x)y + f_4(x)$$
$$(g_1(x)y + g_2(x))\, y' = f_1(x)y^3 + f_2(x)y^2 + f_3(x)y + f_4(x)$$

which are equivalent under the transformation:

$$y \to \frac{g_1(x) - g_2(x)y}{g_1(x)y}$$

Two Abel equations:

$$y' = f_3y^3 + f_2y^2 + f_1y + f_0, u' = \tilde{f}_3u^3 + \tilde{f}_2u^2 + \tilde{f}_1u + \tilde{f}_0$$

are said to be equivalent, if they can be transformed to each other by[Kam13]:

$$\{x = F(t), \quad y = P(t)u + Q(t)\}$$

In order to find the equivalence relationship between a solvable Abel equation and a new derived Abel equation, we use the invariant method.
Given Abel equation:

$$y' = f_3 y^3 + f_2 y^2 + f_1 y + f_0$$

its **relative invariants** are calculated by[Che06]:

$$s_3 \equiv f_0 f_3^2 + \frac{1}{3}\left(\frac{2f_2^3}{9} - f_1 f_2 f_3 + f_3 f_2' - f_2 f_3'\right)$$
$$s_{2m+1} \equiv f_3 s_{2m-1}' - (2m-1) s_{2m-1}\left(f_3' + f_1 f_3 - \frac{f_2^2}{3}\right)$$

and its **absolute invariants** are calculated by:

$$I_1 = \frac{s_5^3}{s_3^5}, I_2 = \frac{s_7 s_3}{s_5^2}, I_3 = \frac{s_9}{s_3^3}, \text{ etc } \dots$$

Two Abel equations:

$$y' = f_3 y^3 + f_2 y^2 + f_1 y + f_0, u' = \tilde{f}_3 u^3 + \tilde{f}_2 u^2 + \tilde{f}_1 u + \tilde{f}_0$$

are equivalent, if:

$$0 = \frac{\tilde{s_5}^3}{\tilde{s_3}^5} - \left.\frac{s_5^3}{s_3^5}\right|_{x=F(t)} \qquad 0 = \frac{\tilde{s_5}\tilde{s_7}}{\tilde{s_3}^4} - \left.\frac{s_5 s_7}{s_3^4}\right|_{x=F(t)}$$

has a common solutions $F(t)$. This $F(t)$ also guarantees all other invariant to have the same value. And by computing $F(t)$, the $P(t), Q(t)$ in linear transformation are given by:

$$P(t) = \left.\frac{F' \tilde{f_3}^2 s_3}{f_3^2 \tilde{s_3}}\right|_{x=F(t)} \qquad Q(t) = \left.\frac{F' \tilde{f_2}\tilde{f_3} s_3 - f_2 f_3 \tilde{s_3}}{3 f_3^2 \tilde{s_3}}\right|_{x=F(t)}$$

During 2000 to 2005, Maple research scientists generalized all solvable classes of Abel differential equation into three classes: Abel inverse linear(AIL), Abel inverse Riccati(AIR), Abel inverse Abel(AIA)[CTR00] The Abel inverse Riccati equation, is generally of the form:

$$\frac{dy}{dx} = \frac{a_1 y^3 + a_2 y^2 + a_3 y + a_4}{\left(b_1 x^2 + b_2 x + b_3\right) y + c_1 x^2 + c_2 x + c_3}$$

and it becomes Riccati equation by switching variables $x \to y, y \to x$. It is generally solvable in terms of special functions[CT05]. It represents all non-Liouvillian integrable case of Abel equations.

In [CTR00], author proposed method to match rational Abel equations with the known solvable AIL, AIR classes through the calculation of absolute invariants. In [CCT04][Kov86], author proposed method for finding both the Liouvillian and non-Liouvillian solutions in terms of special functions of the rational second order linear equation. So do the Riccati equation as well since they are equivalent.

2 Equivalence through Abel equation

We consider the differential equation of the form:

$$y' = \frac{(c_1(y-\rho_1)^m + c_2(y-\rho_2)^m)(y-\rho_1)(y-\rho_2)F'(x)}{(a_1F(x)^2 + a_2F(x) + a_3)(y-\rho_1)^m + (b_1F(x)^2 + b_2F(x) + b_3)(y-\rho_2)^m} \tag{1}$$

where $m \in \mathbb{Z}^+$, $F(x)$ is arbitrary rational function in terms of x. It can be checked that, after transformation:

$$u = \frac{(y-\rho_1)^m}{(y-\rho_2)^m}, t = F(x)$$

(1) can be transformed back to the solvable Abel Inverse Riccati equation. In reality cases, after expansion and collecting the coefficients of the polynomials, it will be hard to determine the exact form of the $F(x)$. However, by a direct transformation in terms of y:

$$u = \frac{(y-\rho_1)^m}{(y-\rho_2)^m}$$

the equation will be transformed to rational Abel equation. And finding the linear transformation that yields this equation to AIR equation can be done by the Maple software.
So the key points is to determine ρ_1, ρ_2, m in the transformation. It can be checked that, all cases of (1) can be writen as the rational differential equation of the form(we don't consider the cases that are solvable in existing Maple dsolve function's framework, such as seperable cases):

$$y' = \frac{M(x)N(y)}{P(x,y)} \tag{2}$$

where $M, N, P \in \mathbb{R}[x,y]$, and the degree of y in $N(y)$ is 1 or 2 degree higher then the degree of y in $P(x,y)$.
The symbolic computation method is presented as follows:

- For equation satisfies the form of (2), try to find all solvable distinct roots of $N(y)$ denoted as $\rho_1, \rho_2 \cdots \rho_k$ using `solve` function in Maple.
- Find the highest degree of the y in $P(x,y)$, denoted as m.

- For each combination of ρ_i, ρ_j in all obtained roots, apply transformation:

$$u = \frac{(y-\rho_i)^m}{(y-\rho_j)^m}$$

  to see whether it can reduce the equation to Abel equation.

- If successful, try to solve the Abel equation using the existing module in dsolve function.

Example. Consider the equation which can't be solved directly using Maple 2023:

```
>(16 - u(t)**4)*(t**5 - t) = diff(u(t), t)*((-t**2 + 1)**4*u(t)**2
+ (t**3 + t)**2*u(t) + 4*(-t**2 + 1)**4)
```

This equation satisfies the form (2), so we do a test following the procedure. The four roots of the $16-u^4$ are $2I, -2I, 2, -2$. By combining the two real roots $2, -2$ and apply the transformation:

```
>simplify(expand(ChangeVariables(t = x, u(t) = 2*(sqrt(y(x)) - 1)/(sqrt(y(x))
+ 1), (16 - u(t)**4)*(t**5 - t) = diff(u(t), t)*((-t**2 + 1)**4*u(t)**2
+ (t**3 + t)**2*u(t) + 4*(-t**2 + 1)**4))), symbolic)
```

we will obtain:

```
>8*(x**5 - x)*(y(x)**2 + y(x)) = ((x**4 - 3/4*x**2 + 1/4)*(x**4 - 3*x**2
+ 4)*y(x) + x**8 - (17*x**6)/4 + (11*x**4)/2 - (17*x**2)/4 + 1)*diff(y(x),
x)
```

which is a rational Abel equation. And the solution is given by:

```
>c- 2*(((x**2 + 1)**2*sqrt(65) + 33*x**4 - 62*x**2 + 33)*(y(x) + 1)*
hypergeom([3*I*sqrt(7)/32 - sqrt(65)/32 + 15/16, -3*I*sqrt(7)/32 -
sqrt(65)/32 + 15/16], [15/8], y(x) + 1) - 28*hypergeom([3*I*sqrt(7)/32
- sqrt(65)/32 - 1/16, -3*I*sqrt(7)/32 - sqrt(65)/32 - 1/16], [7/8],
y(x) + 1)*(x**2 + 1)**2)/((y(x) + 1)**(1/8)*(2*((x**4 - 25/4*x**2 +
1)*sqrt(65) - 33*x**4 + (313*x**2)/4 - 33)*(y(x) + 1)*
hypergeom([3*I*sqrt(7)/32 - sqrt(65)/32 + 17/16, -3*I*sqrt(7)/32 -
sqrt(65)/32 + 17/16], [17/8], y(x) + 1) - 9*x**2*hypergeom([3*I*sqrt(7)/32
- sqrt(65)/32 + 1/16, -3*I*sqrt(7)/32 - sqrt(65)/32 + 1/16], [9/8],
y(x) + 1)*(-33 + sqrt(65)))) = 0
```

3 Equivalence through Riccati equation

we consider the following rational equation:

$$y' = \frac{A(x)(y-\rho_1)^{2m} + B(x)(y-\rho_1)^m(y-\rho_2)^m + C(x)(y-\rho_2)^{2m}}{P(x)((y-\rho_1)(y-\rho_2))^{m-1}}$$

where $A(x), B(x), C(x), P(x) \in \mathbb{R}[x]$, and $m \in \mathbb{Z}^+$. It can be checked that, after a simple transformation in terms of y

$$u = \frac{(y-\rho_1)^m}{(y-\rho_2)^m}$$

that the equation is transformed into a rational Riccati equation. And the integrability can be determined by Maple using existing algorithm.
Whenever we have a differential equation of the form:

$$y' = \frac{M(x, y)}{P(x)((y - \rho_1)(y - \rho_2))^{m-1}} \tag{3}$$

in which the degree of y in $M(x, y)$ is no more then $2m$, we try to apply the transformation, to see whether it can reduce the equation to Riccati equation.
Example. Consider the following equation:

```
>2*(u(t) + 1)*(t + 1)**5*diff(u(t), t)*(u(t) - 1) = (2*t**3 + 6*t)*u(t)**4
+ (-24*t**2 - 8)*u(t)**3 + (12*t**3 + 36*t)*u(t)**2 + (-24*t**2 - 8)*u(t)
+ 2*t**3 + 6*t
```

This equation is of the form (3), by applying transformation:

```
>simplify(expand(ChangeVariables(t = x, u(t) = (y(x)**(1/2) - 1)/(y(x)**(1/2)
+ 1), 2*(u(t) + 1)*(t + 1)**5*diff(u(t), t)*(u(t) - 1) = (2*t**3 +
6*t)*u(t)**4 + (-24*t**2 - 8)*u(t)**3 + (12*t**3 + 36*t)*u(t)**2 +
(-24*t**2 - 8)*u(t) + 2*t**3 + 6*t)), symbolic)
```

It is reduced to rational Riccati equation:

```
diff(y(x), x)*(x + 1)**5 = -2*(x - 1)**3*y(x)**2 - 2*(x + 1)**3
```

which has non-Liouvillian solution:

```
y(x) = sqrt(x**2 - 1)*((c*x + c)*BesselY(-1/5, (2*(x - 1)**2*sqrt(x**2
- 1))/(5*(x + 1)**3)) + (x + 1)*BesselJ(-1/5, (2*(x - 1)**2*sqrt(x**2
- 1))/(5*(x + 1)**3)))/((x - 1)**2*(c*BesselY(4/5, (2*(x - 1)**2*sqrt(x**2
- 1))/(5*(x + 1)**3)) + BesselJ(4/5, (2*(x - 1)**2*sqrt(x**2 - 1))/(5*(x
+ 1)**3))))
```

4 Reference

References

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