Finding equivalence transformation that transforms a rational equation to Riccati equation

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Abstract. The rational differential equation, which can be viewed as a planar autonomous system, remains an important research interest in the study of dynamical system, physics. And the existence of its first integral is closely related to the study of Hilbert 16th problem of such system[Hil00], which aims to find out the connection between the number of limit cycles and the polynomial degree of the system. Unfortunately, first integral in general is difficult to find, especially when the first integral are non-Liouvillian first integral expressed by special function. In this paper, we proposed a symbolic computation method that can be implemented by the Maple language, to tell whether a rational differential equation is equivalent to a Riccati equation under a rational transformation in terms of y. And as far as we know, the tools tackling the non-Liouvillian first integral of Riccati equation, which is equivalent to second order linear equation has been abundant. So this method provides new approach to obtain non-Liouvillian integral of rational differential equation.

Keywords: rational differential equation· Riccati equation · non-Liouvillian first integral · symbolic computation

1 Review

The rational differential equation, which can be viewed as a planar autonomous system, of the form:

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

where $M(x,y), N(x,y) \in \mathbb{R}[x,y]$ remains an important research interest in the study of dynamical system, physics etc[GGG06].And the existence of its first integral is closely related to the study of Hilbert 16th problem of such system[Hil00], which aims to find out the connection between the number of limit cycles and the polynomial degree of the system. The methods to track the first integrals of rational differential equation include the Darboux theory of integrability[Dar78], the use of Lie symmetries[CGG01], the use of Lax pairs[Lax68], the compatibility analysis[SW88] and many more, see[Gor01][Kam13] for an overview.
However, there is a restriction to all the methods mentioned above: they are

not applicable to non-liouvillian first integrals. If a first integral contains special, non-liouvillian functions (e.g. the Bessel or the hypergeometric ones), it cannot be found with the above methods. Such solutions appear when a planar autonomous system yields Riccati equation.
A Riccati equation is a nonlinear first order equation of the form:

$$y' = P(x)y^2 + Q(x)y + R(x), P(x) \neq 0$$

Riccati equation is converted to second order linear equation by transformation:

$$y \to -\frac{y'}{P(x)y}$$

And generally speaking, second order linear equation leads to non-Liouvillian functions as solutions.[Kam13]
There have been many works concerning the special functions solutions of second order linear equation, [CCT04] deals with hypergeometric solutions of linear equation, [DvHK08][VHY10] deals with Bessel functions solutions of second order LODE. These methods have been implemented in the computer algebra system Maple. Maple is a powerful tool to determine the integrability of rational second order LODE. So our goal is to compute possible transformation yields a rational differential equation to Riccati equation, therefore to second order linear equation, from which the non-Liouvillian integral appears.
The idea to relate a rational equation to a second order linear equation is actually not new. In[GGG06][SH06], author states the general form for a linearizable system. That is, a Riccati equation, after a rational transformation $y \to \frac{A(x,y)}{B(x,y)}$, where $A(x,y), B(x,y)$ are polynomials. After the transformation, the structure of the differential equation will become quite complicated and not easy to reduce to the original riccati equation at least from direct observation. Hence, in this paper, we would like to state a method to reveal the transformation $y \to \frac{A(x,y)}{B(x,y)}$ and the equivalent Riccati equation based on the given equation's structure. The paper is organized as follows: in the second and third section we first analyze some specific type of transformation which are easy to discover. We give methods to reveal these transformation and the equivalent Riccati equation. While these methods cannot work, we would use the method in forth section, to figure out whether a general equivalent transformation exist for the given equation.

2 Symbolic computation method for $y \to \frac{A(y)}{B(y)}$

For a rational Riccati equation.:

$$y'T(x) = P(x)y^2 + Q(x)y + R(x)$$

Where $P(x), Q(x), R(x), T(x) \in \mathbb{R}[x], P(x) \neq 0$.The transformation attracts our interest firstly is $y \to P(y)$, where $P(y) = \frac{A(y)}{B(y)}$, $A(y), B(y) \in \mathbb{R}[y]$ is nonconstant arbitrary rational function. After the transformation, the equation becomes:

$$y' = \frac{P(x)A(y)^2 + Q(x)A(y)B(y) + R(x)B(y)^2}{(A'(y)B(y) - A(y)B'(y))T(x)}$$

After expansion and collecting the coefficients of the polynomials, the equation roughly has the form:

$$y' = \frac{P(x, y)}{M(x)N(y)} \tag{1}$$

where the degree of y in $P(x, y)$ is an even number, and larger then the degree of y in $N(y)$.

The aim of this paper is to find out transformation $y \to P^{-1}(y)$ where P is rational function that could reduce the equation of form (1) to Riccati equation whenever possible. Our computation is based on the computer algebra systm Maple. The algorithm is presented as follows.

- find the degree of y in $P(x, y)$, denoted as A, and set:

$$m = \frac{A}{2}$$

- setting two polynomials:

$$A(y) = \sum_{i=0}^{m} a_i y^i$$

$$B(y) = \sum_{i=0}^{m} b_i y^i$$

- compute $A'(y)B(y) - B'(y)A(y)$, and equating the ratio of coefficients to the ratio of coefficients of polynomial $N(y)$, to reduce the parameters in a_i, b_i.
- Express the coefficients of $A(y), B(y)$ using one of the constraints obtain in step 3, and rewrite the polynomial $P(x, y)$ in the ascending order of x:

$$P(x, y) = \sum_{k=0}^{K} x^k p_k(y)$$

- Try to express $p_k(y)$ as the linear combination of $A(y)^2, A(y)B(y), B(y)^2$, that is:

$$p_k(y) = z_1^k A(y)^2 + z_2^k A(y)B(y) + z_3^k B(y)^2$$

  Where $z_i^k \in \mathbb{R}$. Equating the coefficients, to find any possible values for $z_1^k, z_2^k, z_3^k, a_i, b_i$ using `solve` function in Maple. If a set of values obtained, choose one numerical value and enter next step. If no solutions are found, use another constraint on a_i, b_i in step 3 and compute again.
- The equivalent Riccati equation is written as:

$$M(x)y' = \alpha((\sum_{k=0}^{K} z_1^k x^k)y^2 + (\sum_{k=0}^{K} z_2^k x^k)y + (\sum_{k=0}^{K} z_3^k x^k))$$

where $\alpha \in \mathbb{R}$ and:

$$\alpha = \frac{A'(y)B(y) - B'(y)A(y)}{N(y)}$$

Remark. It's important to note the reason for the first step. Since the maximum degree of y in $A(y)^2, A(y)B(y), B(y)^2$ should be either the degree of $A(y)^2$ or $B(y)^2$. When the maximum degree term is eliminated in $P(x,y) = P(x)A(y)^2 + Q(x)A(y)B(y) + R(x)B(y)^2$, it can be checked that the equation itself is Liouville integrable and not necessary to be solved by a reduction to Riccati equation.

Example. Consider equation:

$$y' = \frac{(y^2-1)^4x^2 + (y^3+y)^2x + 4(y^2-1)^4}{(16-x^4)(y^5-y)}$$

This equation is of the form (1), so we use the algorithm to compute the possible equivalence to Riccati equation. Notice that $A = 8$, so $m = 4$, and we set:

$$A(y) = a_1y^4 + a_2y^3 + a_3y^2 + a_4y + a_5$$

$$B(y) = b_1y^4 + b_2y^3 + b_3y^2 + b_4y + b_5$$

Compute $A'(y)B(y) - B'(y)A(y)$:

$$y^6(a_1b_2 - a_2b_1) + y^5(2a_1b_3 - 2a_3b_1) + y^4(3a_1b_4 + a_2b_3 - a_3b_2 - 3a_4b_1) + y^3(4a_1b_5 + 2a_2b_4 - 2a_4b_2 - 4a_5b_1)$$
$$+ y^2(3a_2b_5 + a_3b_4 - a_4b_3 - 3a_5b_2) + y(2a_3b_5 - 2a_5b_3) + a_4b_5 - a_5b_4$$

And compare the coefficients with $N(y) = y^5 - y$, we have:

$$a_1b_2 - a_2b_1 = 0$$
$$a_1b_3 - a_3b_1 = a_5b_3 - a_3b_5$$
$$3a_1b_4 + a_2b_3 - a_3b_2 - 3a_4b_1 = 0$$
$$4a_1b_5 + 2a_2b_4 - 2a_4b_2 - 4a_5b_1 = 0$$
$$3a_2b_5 + a_3b_4 - a_4b_3 - 3a_5b_2 = 0$$
$$a_4b_5 - a_5b_4 = 0$$

Solve the equations using Maple, we obtain the following sets of non-trivial solutions:

$$\left\{a_1 = \frac{a_5b_1}{b_5}, a_2 = \frac{a_5b_2}{b_5}, a_3 = \frac{a_5b_3}{b_5}, a_4 = \frac{a_5b_4}{b_5}, a_5 = a_5, b_1 = b_1, b_2 = b_2, b_3 = b_3, b_4 = b_4, b_5 = b_5\right\} \tag{2}$$

$$\{a_1 = a_5, a_2 = 0, a_3 = a_3, a_4 = 0, a_5 = a_5, b_1 = b_5, b_2 = 0, b_3 = b_3, b_4 = 0, b_5 = b_5\} \tag{3}$$

$$\left\{a_1 = a_1, a_2 = a_2, a_3 = a_3, a_4 = a_4, a_5 = 0, b_1 = b_1, b_2 = \frac{a_2b_1}{a_1}, b_3 = \frac{a_3b_1}{a_1}, b_4 = \frac{a_4b_1}{a_1}, b_5 = 0\right\} \tag{4}$$

$$\left\{a_1 = 0, a_2 = a_2, a_3 = a_3, a_4 = a_4, a_5 = 0, b_1 = 0, b_2 = \frac{a_2 b_3}{a_3}, b_3 = b_3, b_4 = \frac{a_4 b_3}{a_3}, b_5 = 0\right\} \tag{5}$$

$$\{a_1 = 0, a_2 = a_2, a_3 = 0, a_4 = a_4, a_5 = 0, b_1 = 0, b_2 = b_2, b_3 = 0, b_4 = b_4, b_5 = 0\} \tag{6}$$

Choose constraint (3), then $A(y), B(y)$ becomes:

$$A(y) = a_5 y^4 + a_3 y^2 + a_5$$

$$B(y) = b_5 y^4 + b_3 y^2 + b_5$$

Now consider $P(x, y)$, the coefficients of x^k are $(y^2 - 1)^4, (y^3 + y)^2, 4(y^2 - 1)^4$ respectively. Compare $z_1^k A(y)^2 + z_2^k A(y)B(y) + z_3^k B(y)^2$ with these polynomials respectively, and equating the coefficients, we obtain:

$$\left\{z_1^2 = \frac{b_3^2 + 4b_3 b_5 + 4b_5^2}{a_3^2 b_5^2 - 2a_3 a_5 b_3 b_5 + a_5^2 b_3^2}, z_2^2 = -\frac{2\left(a_3 b_3 + 2a_3 b_5 + 2a_5 b_3 + 4a_5 b_5\right)}{\left(a_3 b_5 - a_5 b_3\right)^2}, z_3^2 = \frac{a_3^2 + 4a_3 a_5 + 4a_5^2}{a_3^2 b_5^2 - 2a_3 a_5 b_3 b_5 + a_5^2 b_3^2}\right\}$$

$$\left\{z_1^1 = -\frac{\left(b_3 - 2b_5\right) b_5}{a_3^2 b_5^2 - 2a_3 a_5 b_3 b_5 + a_5^2 b_3^2}, z_2^1 = \frac{a_3 b_5 + a_5 b_3 - 4a_5 b_5}{\left(a_3 b_5 - a_5 b_3\right)^2}, z_3^1 = -\frac{a_5\left(a_3 - 2a_5\right)}{a_3^2 b_5^2 - 2a_3 a_5 b_3 b_5 + a_5^2 b_3^2}\right\}$$

$$\left\{z_1^0 = \frac{4\left(b_3^2 + 4b_3 b_5 + 4b_5^2\right)}{a_3^2 b_5^2 - 2a_3 a_5 b_3 b_5 + a_5^2 b_3^2}, z_2^0 = -\frac{8\left(a_3 b_3 + 2a_3 b_5 + 2a_5 b_3 + 4a_5 b_5\right)}{\left(a_3 b_5 - a_5 b_3\right)^2}, z_3^0 = \frac{4\left(a_3^2 + 4a_3 a_5 + 4a_5^2\right)}{a_3^2 b_5^2 - 2a_3 a_5 b_3 b_5 + a_5^2 b_3^2}\right\}$$

And a_3, a_5, b_3, b_5 can be arbitrary constants, so we randomly choose:

$$a_3 = -6, a_5 = 1, b_3 = -2, b_5 = 1$$

Compute z_i^k and $\alpha = \frac{A'(y)B(y) - B'(y)A(y)}{N(y)}$, and derive the equivalent Riccati equation as follows:

$$\begin{aligned}(16 - x^4)y' &= \alpha((\sum_{k=0}^{2} z_1^k x^k)y^2 + (\sum_{k=0}^{2} z_2^k x^k)y + (\sum_{k=0}^{2} z_3^k x^k)) \\ &= 2y^2 x - 6yx + 8x^2 + 4x + 32\end{aligned}$$

Use maple to verify this equation has non-Liouvillian first integral in terms of hypergeometric function. Substitute the y in the first integral with $\frac{A(y)}{B(y)}$ will be the solution for the original equation.

3 Symbolic computation method for $y \to \frac{A(x,y)}{B(x)}$

In the paper [GGG08], the author proposed an algorithm for finding the inverse integrating factor of a linearizable equation. He showed that the integrating factor of rational equation can be obtained by the solutions of equivalent second order linear equation. Furthermore, he showed that the integrating factor is a polynomial in terms of y when the equivalent transformation is of the form $y \to \frac{A(x,y)}{B(x)}$, where $A(x,y), B(x) \in \mathbb{R}[x,y]$. He treated the coefficient function of y as undetermined, and proved that they satisfy a linear equation system where the highest order is 2. However, solving these equation, although possible, will be quite time consuming.

In this section we will propose a more efficient algorithm for finding equivalence between rational equation and Riccati equation under transformation $y \to \frac{A(x,y)}{B(x)}$. First, we point out a fact for simplifying this problem.

Theorem. Any transformation $y \to P(x)A(x,y) + Q(x)$ that transforms a Riccati equation to a more general rational equation could be reversed to transform the rational equation to Riccati equation simply by $y \to A^{-1}(x,y)$.

Proof. the transformation $y \to P(x)A(x,y) + Q(x)$ can be decomposed into two transformation. $y \to P(x)y + Q(x)$ and subsequently $y \to A(x,y)$. The first transformation is called linear transformation, which does not change the structure of a Riccati equation. So only $A(x,y)$ will be possible to change the structure of Riccati equation.

Therefore, we can eliminate $B(x)$ in the transformation and consider $A(x,y)$ only.

Consider the rational Riccati equation:

$$y'T(x) = P(x)y^2 + Q(x)y + R(x)$$

Where $P(x), Q(x), R(x), T(x) \in \mathbb{R}[x], P(x) \neq 0$. Apply the transformation $y \to A(x,y)$, the equation is transformed to the form:

$$\frac{\partial A(x,y)}{\partial y}T(x)y' = A(x,y)^2P(x) + A(x,y)Q(x) + R(x) - T(x)\frac{\partial A(x,y)}{\partial x}$$

Based on the structure of the derived equation, the algorithm for obtaining $A(x,y)$ and the corresponding Riccati equation is as follows:

- For a rational differential equation $y' = \frac{M(x,y)}{N(x,y)}$, Try to factor $N(x,y)$ into $N_1(x)N_2(x,y)$, where $N_2(x,y)$ does not has a factor of the form $x - a$
- Integrate $N_2(x,y)$ with respect to y, and use the simplest form, which means without adding $F(x)$ to the result to be $A(x,y)$

$$A(x,y) = \int N_2(x,y)\mathrm{d}y$$

- For the Polynomial $M(x,y)+N_1(x)\frac{\partial A(x,y)}{\partial x}$, treat y as the variable, and apply long division with the divisor $A(x,y)$, if the remainder A_3 contains only x, then go on to divide the quotient with divisor $A(x,y)$, if the remainder A_2 and the quotient A_1 contain only x, then the corresponding Riccati equation under transformation $y \to A(x,y)$ is:

$$y'N_1(x) = A_1(x)y^2 + A_2(x)y + A_3(x)$$

This method only contains basic integration and polynomial calculations, and will be much easier to carry out in the computer software like maple than method in [GGG08].

Example. Consider the differential equation:

$$y' = \frac{x^3y^4 + 2x^2y^3 + (x^3 + 3x^2 + 2x - 1)y^2 + (x^2 + 4x + 1)y + 2x^4 + 5x^2 + 3x + 3}{(2xy + 1)(x^2 + 1)}$$

First verify that $N_1(x) = x^2 + 1, N_2(x, y) = 2xy + 1$, so $A(x, y) = xy^2 + y$.And:

$$M(x,y)+N_1(x)\frac{\partial A(x,y)}{\partial x} = x^3y^4+2x^2y^3+(x^3+4x^2+2x)y^2+(x^2+4x+1)y+2x^4+5x^2+3x+3$$

Apply long division with the divisor $A(x, y)$ twice, we will obtain $A_1(x) = x, A_2(x) = x^2 + 4x + 1, A_3(x) = 2x^4 + 5x^2 + 3x + 3$, the equivalent Riccati equation under the transformation $y \to A(x, y)$ reads:

$$(1 + x^2)y' = xy^2 + (x^2 + 4x + 1)y + 2x^4 + 5x^2 + 3x + 3$$

has non-Liouvillian first integral in terms of Airy function.

4 Symbolic computation method for $y \to \frac{A(x,y)}{B(x,y)}$

When none of the methods in the previous section work, we must apply a more general assumption on the equivalence transformation. Assume applying a rational transformation $y \to \frac{A(x,y)}{B(x,y)}$ to a rational Riccati equation:

$$y'T(x) = P(x)y^2 + Q(x)y + R(x)$$

The Riccati equation becomes:

$$y' = \frac{A(x,y)^2P(x) + A(x,y)B(x,y)Q(x) + B(x,y)^2R(x) - T(x)(B(x,y)\frac{\partial}{\partial x}A(x,y) - A(x,y)\frac{\partial}{\partial x}B(x,y))}{(B(x,y)\frac{\partial}{\partial y}A(x,y) - A(x,y)\frac{\partial}{\partial y}B(x,y))T(x)}$$

To match a given equation with the transformed form of a Riccati equation, we propose the following computation method:

– For a rational differential equation $y' = \frac{M(x,y)}{N(x,y)}$, if the degree of y in $M(x, y)$ is $2m, m \in \mathbb{N}$, then we equate:

$$A(x,y) = \sum_{i=1}^{m} a_i(x)y^i$$

$$B(x,y) = \sum_{i=1}^{m} b_i(x)y^i$$

– Factor $N(x, y)$ into $N_1(x)N_2(x, y)$, where $N_2(x, y)$ does not have any factor of the form $x - a$.

- Try to equate(compare coefficients of y and equate them, herein after the same):

$$B(x,y)\frac{\partial}{\partial y}A(x,y)-A(x,y)\frac{\partial}{\partial y}B(x,y)=N_2(x,y)$$

  and solve the $a_i(x), b_i(x)$ in terms of x. Normally in this step various sets of $a_i(x), b_i(x)$ are obtained.
- Choose a specific value of solution $a_i(x), b_i(x)$, and Try to equate:

$$P(x)A(x,y)^2+Q(x)A(x,y)B(x,y)+R(x)B(x,y)^2=M(x,y)+N_1(x)(B(x,y)\frac{\partial}{\partial x}A(x,y)-A(x,y)\frac{\partial}{\partial x}B(x,y))$$

  if solutions of $P(x), Q(x), R(x)$ are found, the equivalent Riccati equation under rational transformation $y \to \frac{A(x,y)}{B(x,y)}$ is:

$$y'N_1(x)=P(x)y^2+Q(x)y+R(x)$$

  If no solutions are found, choose another set of solutions in step 3 and compute again

Example. Consider the differential equation:

$$y'=\frac{(2x^2+6x+4)\,y^4+(3x^3+x+2)\,y^3+(5x^5+8x^4+5x^3-3x^2+x)\,y^2+(3x^6+2x^5+3x^4+4x^3)\,y+2x^8+2x^7+x^6}{(-x^3-x^2-x-1)y^2-2x^3y+x^6+x^4}$$

Compare it with the structure of a linearizable equation, we obtain:

$$M(x,y)=\left(2x^2+6x+4\right)y^4+\left(3x^3+x+2\right)y^3+\left(5x^5+8x^4+5x^3-3x^2+x\right)y^2+\left(3x^6+2x^5+3x^4+4x^3\right)y+2x^8+2x^7+x^6$$

$$N_2(x,y)=(-x^3-x^2-x-1)y^2-2x^3y+x^6+x^4, N_1(x)=1$$

The degree of y in $M(x,y)$ is 4, so:

$$A(x,y)=\sum_{i=1}^{2}a_i(x)y^i$$

$$B(x,y)=\sum_{i=1}^{2}b_i(x)y^i$$

Equate $B(x,y)\frac{\partial}{\partial y}A(x,y)-A(x,y)\frac{\partial}{\partial y}B(x,y)$ and $N_2(x,y)$, the following set of solutions is obtained:

$$\left\{a_0=\frac{x^3}{b_2}, a_1=-\frac{x^3a_2b_2-x^3+xa_2b_2-x^2-x-1}{b_2}, a_2=a_2, b_0=0, b_1=-x^3b_2-xb_2, b_2=b_2\right\}$$

Choose $a_2=b_2=1$, then $A(x,y)=y^2+y(x^2+1)+x^3, B(x,y)=y^2+y(-x^3-x)$.
Compare the coefficients of y on both sides of the equation:

$$P(x)A(x,y)^2+Q(x)A(x,y)B(x,y)+R(x)B(x,y)^2=M(x,y)+N_1(x)(B(x,y)\frac{\partial}{\partial x}A(x,y)-A(x,y)\frac{\partial}{\partial x}B(x,y))$$

we obtain:

$$P(x) = 2x^2 + 2x + 1, Q(x) = 4x + 1, R(x) = 2$$

The equivalent Riccati equation under transformation $y \to \frac{y^2+y(x^2+1)+x^3}{y^2+y(-x^3-x)}$ reads:

$$y' = (2x^2 + 2x + 1)y^2 + (4x + 1)y + 2$$

has non-Liouvillian solutions in terms of hypergeometric function.

5 Conclusion

We propose an efficient method for finding the non-Liouvillian first integral for a specific type of rational equation. Such method will be useful not only for computing the first integral, but also for determining other properties including limit cycles, center focus for the corresponding planar autonomous system.Even though we reduce the type of equation mentioned in this paper to rational Riccati equation that is not solvable, we can succeed in ascertaining the fact that the equation does not have Liouvillian first integral by kovacic algorithm[Kov86]

6 Reference

References

[CCT04] L. Chan and E. S. Cheb-Terrab. Non-liouvillian solutions for second order linear odes. In *International Symposium on Symbolic and Algebraic Computation*, 2004.

[CGG01] J Chavarriga, IA García, and J Giné. On lie's symmetries for planar polynomial differential systems. *Nonlinearity*, 14(4):863, 2001.

[Dar78] Gaston Darboux. Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré. *Bulletin des sciences mathématiques et astronomiques*, 2(1):151–200, 1878.

[DvHK08] Ruben Debeerst, Mark van Hoeij, and Wolfram Koepf. Solving differential equations in terms of bessel functions. In *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*, pages 39–46, 2008.

[GGG06] Héctor Giacomini, Jaume Giné, and Maite Grau. Integrability of planar polynomial differential systems through linear differential equations. *The Rocky Mountain Journal of Mathematics*, pages 457–485, 2006.

[GGG08] Héctor Giacomini, Jaume Giné, and Maite Grau. Linearizable planar differential systems via the inverse integrating factor. *Journal of Physics A: Mathematical and Theoretical*, 41(13):135205, 2008.

[Gor01] Alain Goriely. *Integrability and nonintegrability of dynamical systems*, volume 19. World Scientific, 2001.

[Hil00] David Hilbert. Mathematical problems. *Bulletin of the American Mathematical Society*, 37(4):407–436, 2000.

[Kam13] Erich Kamke. *Differentialgleichungen lösungsmethoden und lösungen.* Springer-Verlag, 2013.

[Kov86] Jerald J Kovacic. An algorithm for solving second order linear homogeneous differential equations. *Journal of Symbolic Computation*, 2(1):3–43, 1986.

[Lax68] Peter D Lax. Integrals of nonlinear equations of evolution and solitary waves. *Communications on pure and applied mathematics*, 21(5):467–490, 1968.

[SH06] Axel Schulze-Halberg. A simple method for constructing non-liouvillian first integrals of autonomous planar systems. *Czechoslovak mathematical journal*, 56:987–999, 2006.

[SW88] J-M Strelcyn and S Wojciechowski. A method of finding integrals for three-dimensional dynamical systems. *Physics Letters A*, 133(4-5):207–212, 1988.

[VHY10] Mark Van Hoeij and Quan Yuan. Finding all bessel type solutions for linear differential equations with rational function coefficients. In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, pages 37–44, 2010.