4. Numerical simulation

We perform numerical simulations based on the fractional Adams-Bashforth-Moulton method [2]. It was shown that for a differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t)),$$

the fractional variant of the one-step Adams-Moulton method is given by (corrector formula)

$$x_{n+1} = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_{n+1}^p),$$

in which $t_i = ih$ with some fixed h, and

$$a_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \ i = 0 \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1} - 2(n-i+1)^{\alpha+1}, \ 1 \le i \le n. \end{cases}$$

We first need to compute the values x_{n+1}^p, given by the generalize one-step Adams-Bashforth method as a predictor formula

$$x_{n+1}^p = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} f(t_i, x_i),$$

where $b_{i,n+1} = (n+1-i)^\alpha - (n-i)^\alpha$. This method is said to be of Predict, Evaluate, Correct, Evaluate ($PECE$) type, because in a concrete implementation, we would start to calculate the predictor x_{n+1}^p, then we evaluate $f(t_{n+1}, x_{n+1}^p)$. Next, we use these quantities to calculate the corrector in x_{n+1} , and finally evaluate $f(t_{n+1}, x_{n+1})$. This result is stored for future use in the next integration step.

To perform an error analysis of the presented method, we first assume that $t_i = ih = \frac{i\mathcal{T}}{N}$ with some $N \in \mathbb{N}$, and we have the following theorem:

Proposition 4.1. [2] *Let* $\alpha > 0$ *and* $\frac{d^\alpha x(t)}{dt^\alpha} \in C^2[0, \mathcal{T}]$ *for some suitable* $\mathcal{T}$*, then,*

$$\max_{0 \le i \le N} |x(t_i) - x_i| = \begin{cases} O(h^2), \ \alpha \ge 1, \\ O(h^{1+\alpha}), \ \alpha < 1, \end{cases}$$

Applying the corrector formula improves the accuracy of its input (the predictor) by a factor of h^α up to order of $O(h^2)$ for which a saturation is reached. Thus by replacing the plain $PECE$ structure by a $P(EC)^\mu E$ method (additional corrector iterations) a corrector iteration is of the form (corrector formula)

$$x_{n+1}^{[l]} = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_{n+1}^{[l-1]}),$$

in which $x_{n+1}^{[l]}$ denotes the approximation after l corrector steps, $x_{n+1}^{[0]} = x_{n+1}^p$ is the predictor, and $x_{n+1} := x_{n+1}^{[\mu]}$ is the final approximation after μ corrector steps. The following theorem provides an error analysis of this method.

Proposition 4.2. [2] *Assume* $\frac{d^\alpha x(t)}{dt^\alpha} \in C^2[0, \mathcal{T}]$ *for some suitable* $\mathcal{T}$*, and* $\alpha > 0$*. Then, the approximation obtained by the* $P(EC)^\mu E$ *method described above satisfies*

$$\max_{0 \leq i \leq N} |x(t_i) - x_i| = O(h^q),$$

where $q = \min\{2, 1 + \mu\alpha\}$.

Now by employing this method for $0 < \alpha < 1$ to the following system

$$\frac{d^\alpha x(t)}{dt^\alpha} = F(t, x(t), y(t)),$$
$$\frac{d^\alpha y(t)}{dt^\alpha} = G(t, x(t), y(t)),$$

we deduce that

$$x_{n+1}^{[l]} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} F(t_i, x_i, y_i, z_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{n+1}, x_{n+1}^{[l-1]}, y_{n+1}^{[l-1]}, z_{n+1}^{[l-1]}),$$

$$y_{n+1}^{[l]} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} G(t_i, x_i, y_i, z_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} G(t_{n+1}, x_{n+1}^{[l-1]}, y_{n+1}^{[l-1]}, z_{n+1}^{[l-1]}),$$

$$x_{n+1}^{[0]} = x_{n+1}^p = x_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} F(t_i, x_i, y_i, z_i),$$

$$y_{n+1}^{[0]} = y_{n+1}^p = y_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} G(t_i, x_i, y_i, z_i),$$

We now consider k (carrying capacity of the environment) as a bifurcation parameter and consider the fix parameter values $r = 0.05, a = 0.8, \mu = 0.8, \alpha = 0.98$.
Our numerical simulations are plotted in Figure 1 for $d = 0.24, \eta = 0.01, \beta = 0.6, k = 1.6$ and Figure 2 for $d = 0.015, \eta = 0.01, \beta = 0.4, k = 5$ show that α has an essential role on the stability behaviour of this system.

The analytical results can be exploited to examine the obtained numerical results. We now consider the set of fixed parameters specified for Figure 1. For $\alpha = 0.98$ we have $arg(\lambda|_{E_1}) = 1.5649$ and $\frac{\alpha\pi}{2} = 1.5394$. Hence, $|arg(\lambda|_{E_1})| > \frac{\alpha\pi}{2}$, which indicates the condition of asymptotic stability of E_1, based on the Theorem 3.5, holds. Further, for $\alpha = 1$ we have $arg(\lambda|_{E_1}) = 1.5649$ and $\frac{\alpha\pi}{2} = 1.5708$, then $|arg(\lambda|_{E_1})| < \frac{\alpha\pi}{2}$. This reveals that the condition of asymptotic stability of E_1 is violated and E_1 becomes unstable. Hence, a stable limit cycle emerges around E_1. These scenarios are depicted in Figure 1 (right) and Figure 1 (left), respectively.
We next consider the set of fixed parameters specified for Figure 2. For $\alpha = 0.98$ we have $arg(\lambda|_{E_1}) = 1.5695$ and $\frac{\alpha\pi}{2} = 1.5394$, then $|arg(\lambda|_{E_1})| > \frac{\alpha\pi}{2}$ Hence, the condition of asymptotic stability of E_1 holds. Also for $\alpha = 1$ we have $arg(\lambda|_{E_1}) = 1.5695$ and $\frac{\alpha\pi}{2} = 1.5708$, then $|arg(\lambda|_{E_1})| < \frac{\alpha\pi}{2}$. Hence, the condition of asymptotic stability of E_1 is violated and E_1 becomes unstable. Hence, a stable limit cycle emerges around

E_1. These scenarios are depicted in Figure 2 (right) and Figure 2 (left), respectively.