Application of the Homotopy Perturbation Method to Linear and Nonlinear Schrödinger Equations

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He's homotopy perturbation method (HPM) is applied to linear and nonlinear Schrödinger equations for obtaining exact solutions. The HPM is used for an analytic treatment of these equations. The results reveal that the HPM is very effective, convenient and quite accurate to such types of partial differential equations.

Key words: Homotopy Perturbation Method; Variational Iteration Method; Schrödinger Equations.

1. Introduction

The homotopy perturbation method (HPM) was firstly proposed by Ji-Huan He [1 – 4]. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a "small parameter". The HPM deforms a difficult problem into a simple problem which can be easily solved. In [5, 6] He gave a very lucid as well as elementary discussion of why the HPM works so well for both linear and nonlinear equations. In [3], a comparison of the HPM and homotopy analysis method was made, revealing that the former is more powerful than the latter. The HPM gives rapidly convergent series to the exact solution if such a solution exists. Recently, many authors applied this method to various problems and demonstrated the efficiency of it to handle nonlinear structures and solve various physics and engineering problems [7, 8].

The Schrödinger equations occur in various areas of physics, including nonlinear optics, hydrodynamics, plasma physics, superconductivity and quantum mechanics. For example, the linear Schrödinger equation discusses the time evolution of a free particle, and the cubic nonlinear Schrödinger equations exhibit solitary type solutions. Many methods are usually used to handle the nonlinear equations such as the inverse scattering method, the tanh method, Hirota bilinear forms, Backlund transformation, variational iteration method (VIM) [9 – 12] and other methods as well. Recently, Wazwaz [12] applied the VIM to establish exact solutions for some initial value problems of linear and nonlinear Schrödinger equations.

There are three main objectives of this paper. The first is to apply the HPM to some linear and nonlinear Schrödinger equations initial value problems for establishing exact solutions. The second is to confirm the power of the HPM in the treatment of linear and nonlinear equations of scientific and engineering problems in a unified manner without requiring any additional restriction. The third is to compare the results obtained by the HPM with that obtained by Wazwaz [12] using the VIM.

2. Basic Ideas of He's Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [1]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function, and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but

N is nonlinear. Therefore (1) can be rewritten as

$$L(u)+N(u)-f(r)=0. \quad (3)$$

By the homotopy technique, we construct a homotopy $(r,p):\Omega\times[0,1]\to R$ which satisfies

$$H(V,p)=(1-p)[L(V)-L(u_0)]+p[A(V)-f(r)] = 0, \quad p\in[0,1], \quad r\in\Omega, \quad (4)$$

where $p\in[0,1]$ is an embedding parameter and u_0 an initial approximation of (1), which satisfies the boundary conditions.

Obviously, from (4) we have

$$H(V,0)=L(V)-L(u_0)=0, \quad (5)$$

$$H(V,1)=A(V)-f(r)=0, \quad (6)$$

and the changing process of p from zero to unity is just that of (r,p) from $u_0(r)$ to $u(r)$.

According to the HPM, we can use the embedding parameter p as a "small parameter", and assume that the solution of (4) can be written as a power series in p:

$$V=V_0+pV_1+p^2V_2+\cdots. \quad (7)$$

Setting $p=1$ results in the approximate solution of (1)

$$u=\lim_{p\to1}V=V_0+V_1+V_2+\cdots. \quad (8)$$

The series in (8) is convergent for most cases, and also the rate of convergence depends on the nonlinear operator $A(V)$ [1].

The HPM eliminates the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

3. Applications

3.1. The Linear Schrödinger Equation

Example 1. Firstly, we consider the linear Schrödinger equation

$$u_t+\mathrm{i}u_{xx}=0, \quad u(x,0)=1+\cosh(2x), \quad (9)$$

where $u(x,t)$ is a complex function and $\mathrm{i}^2=-1$.

According to (4), a homotopy $(x,t,p):\Omega\times[0,1]\to C$ can be constructed as follows:

$$(1-p)(V_t-u_{0,t})+p(V_t+\mathrm{i}V_{xx})=0, \quad p\in[0,1], \quad (x,t)\in\Omega, \quad (10)$$

where $u_0(x,t)=V_0(x,0)=u(x,0)$ and $u_{0,t}=\partial u_0/\partial t$.

We now try to get a solution of (10) in the form

$$V(x,t)=V_0(x,t)+pV_1(x,t)+p^2V_2(x,t)+\cdots. \quad (11)$$

Substituting (11) into (10), and equating the terms with the identical powers of p, yields

$$\begin{aligned} &p^0: V_{0,t}=0,\\ &p^1: V_{1,t}+\mathrm{i}V_{0,xx}=0,\\ &p^2: V_{2,t}+\mathrm{i}V_{1,xx}=0,\\ &\vdots\\ &p^n: V_{n,t}+\mathrm{i}V_{n-1,xx}=0,\\ &n=3,4,5,\cdots, \end{aligned} \quad (12)$$

with the following initial conditions:

$$V_i(x,0)=\begin{cases}1+\cosh(2x), & i=0,\\ 0, & i=1,2,3,\cdots.\end{cases} \quad (13)$$

The solution of the system (12), with the initial conditions (13), can be easily obtained as follows:

$$\begin{aligned} V_0(x,t)&=1+\cosh(2x),\\ V_1(x,t)&=-4\mathrm{i}t\cosh(2x),\\ V_2(x,t)&=-8t^2\cosh(2x),\\ V_3(x,t)&=\frac{32}{3}\mathrm{i}t^3\cosh(2x),\\ V_4(x,t)&=\frac{32}{3}t^4\cosh(2x),\\ V_5(x,t)&=-\frac{128}{15}\mathrm{i}t^5\cosh(2x). \end{aligned} \quad (14)$$

In this manner the other components can be easily obtained. Substituting (14) into (8) yields

$$u(x,t)=(1+\cosh(2x))\Big(1-4\mathrm{i}t-8t^2+\frac{32}{3}\mathrm{i}t^3 +\frac{32}{3}t^4-\frac{128}{15}\mathrm{i}t^5-\cdots\Big). \quad (15)$$

Consequently, the exact solution of (9)

$$u(x,t)=1+\cosh(2x)\mathrm{e}^{-4\mathrm{i}t}, \quad (16)$$

is readily obtained upon using the Taylor series expansion of $\mathrm{e}^{-4\mathrm{i}t}$.

Remark 1. The solution $u(x,t)=1+\cosh(2x)\mathrm{e}^{-4\mathrm{i}t}$, obtained by Wazwaz [12] for (9) using the VIM, is incorrect. Moreover, it is the solution of the equation $u_t - \mathrm{i}u_{xx} = 0$, $u(x,0) = 1 + \cosh(2x)$.

Example 2. Secondly, we consider the linear Schrödinger equation

$$u_t + \mathrm{i}u_{xx} = 0, \quad u(x,0) = \mathrm{e}^{3\mathrm{i}x}. \tag{17}$$

In the same manner as done in example 1, we obtain the system (12) with the following conditions:

$$V_1(x,0) = \begin{cases} \mathrm{e}^{3\mathrm{i}x}, & i = 0, \\ 0, & i = 1,2,3,\cdots. \end{cases} \tag{18}$$

Solving the system (12), with the initial conditions (18), yields

$$\begin{aligned} V_0(x,t) &= \mathrm{e}^{3\mathrm{i}x}, \\ V_1(x,t) &= 9\mathrm{i}t\mathrm{e}^{3\mathrm{i}x}, \\ V_2(x,t) &= -\frac{81}{2}t^2\mathrm{e}^{3\mathrm{i}x}, \\ V_3(x,t) &= -\frac{243}{2}\mathrm{i}t^3\mathrm{e}^{3\mathrm{i}x}, \\ V_4(x,t) &= \frac{2187}{8}t^4\mathrm{e}^{3\mathrm{i}x}, \\ V_5(x,t) &= \frac{19683}{40}\mathrm{i}t^5\mathrm{e}^{3\mathrm{i}x}. \end{aligned} \tag{19}$$

In this manner the further components can be simply obtained.

Substituting (19) into (8) yields

$$\begin{aligned} u(x,t) = \mathrm{e}^{3\mathrm{i}x}\Big\{&1 + 9\mathrm{i}t - \frac{81}{2}t^2 - \frac{243}{2}\mathrm{i}t^3 \\ &+ \frac{2187}{8}t^4 + \frac{19683}{40}\mathrm{i}t^5 - \cdots\Big\}. \end{aligned} \tag{20}$$

The exact solution of (17),

$$u(x,t) = \mathrm{e}^{3\mathrm{i}(x+3t)}, \tag{21}$$

follows immediately upon using the Taylor series expansion of $\mathrm{e}^{9\mathrm{i}t}$.

Remark 2. The solution $u(x,t) = \mathrm{e}^{3\mathrm{i}(x+3t)}$, obtained by Wazwaz [12] for (17) using the VIM, is incorrect. Moreover, this solution is the exact solution of the equation $u_t - \mathrm{i}u_{xx} = 0$, $u(x,0) = \mathrm{e}^{3\mathrm{i}x}$.

3.2. The Nonlinear Schrödinger Equation

Example 3. We first consider the cubic nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + m|u|^2u = 0, \quad u(x,0) = \mathrm{e}^{n\,\mathrm{i}x}, \tag{22}$$

where m and n are constants.

We construct the homotopy $(x,t,p) : \Omega \times [0,1] \to C$ which satisfies

$$\begin{aligned} &(1-p)(\mathrm{i}V_t - \mathrm{i}u_{0,t}) + p(\mathrm{i}V_t + V_{xx} + m|V|^2V) = 0, \\ &p \in [0,1], \quad (x,t) \in \Omega, \end{aligned} \tag{23}$$

or

$$\begin{aligned} &(1-p)(\mathrm{i}V_t - \mathrm{i}u_{0,t}) + p(\mathrm{i}V_t + V_{xx} + mV^2\bar{V}) = 0, \\ &p \in [0,1], \quad (x,t) \in \Omega, \end{aligned} \tag{24}$$

where $u_0(x,t) = V_0(x,0) = u(x,0)$, $|V|^2 = V\bar{V}$ and $\bar{V}$ is the conjugate of V.

Suppose that the series solution of (24) and its conjugate have the following forms:

$$V = V_0(x,t) + pV_1(x,t) + p^2V_2(x,t) + \cdots, \tag{25}$$

$$\bar{V} = \bar{V}_0(x,t) + p\bar{V}_1(x,t) + p^2\bar{V}_2(x,t) + \cdots. \tag{26}$$

Substituting (25) and (26) into (24), and arranging the coefficients of "p" powers, we have

$$\begin{aligned} &p^0 : \mathrm{i}V_{0,t} = 0, \\ &p^1 : \mathrm{i}V_{1,t} + V_{0,xx} + mV_0^2\bar{V}_0 = 0, \\ &p^2 : \mathrm{i}V_{2,t} + V_{1,xx} + mV_0^2\bar{V}_1 + 2mV_0V_1\bar{V}_0 = 0, \\ &p^3 : \mathrm{i}V_{3,t} + V_{2,xx} + mV_0^2\bar{V}_2 + m\bar{V}_0(V_1^2 + 2V_0V_2) \\ &\quad + 2mV_0V_1\bar{V}_1 = 0, \\ &p^4 : \mathrm{i}V_{4,t} + V_{3,xx} + mV_0^2\bar{V}_3 + 2m\bar{V}_0(V_0V_3 + V_1V_2) \\ &\quad + m\bar{V}_1(V_1^2 + 2V_0V_2) + 2mV_0V_1\bar{V}_2 = 0, \\ &p^5 : \mathrm{i}V_{5,t} + V_{4,xx} + mV_0^2\bar{V}_4 + m\bar{V}_0(V_2^2 + 2V_0V_4 + 2V_1V_3) \\ &\quad + 2m\bar{V}_1(V_0V_3 + V_1V_2) + m\bar{V}_2(V_1^2 + 2V_0V_2) \\ &\quad + 2mV_0V_1\bar{V}_3 = 0, \end{aligned} \tag{27}$$

with the following initial conditions:

$$V_i(x,0) = \begin{cases} \mathrm{e}^{n\,\mathrm{i}x}, & i = 0, \\ 0, & i = 1,2,3,\cdots. \end{cases} \tag{28}$$

The solution of the system (27), with the initial conditions (28), can be easily obtained as follows:

$$V_0(x,t) = \mathrm{e}^{n\,\mathrm{i}x},$$

$$V_1(x,t) = \mathrm{i}(m-n^2)t\,\mathrm{e}^{n\,\mathrm{i}x},$$
$$V_2(x,t) = -\frac{1}{2}(m-n^2)^2t^2\mathrm{e}^{n\,\mathrm{i}x},$$
$$V_3(x,t) = -\frac{\mathrm{i}}{6}(m-n^2)^3t^3\mathrm{e}^{n\,\mathrm{i}x},$$
$$V_4(x,t) = \frac{1}{24}(m-n^2)^4t^4\mathrm{e}^{n\,\mathrm{i}x},$$
$$V_5(x,t) = \frac{\mathrm{i}}{120}(m-n^2)^5t^5\mathrm{e}^{n\,\mathrm{i}x}. \quad (29)$$

The other components can also be easily obtained. Substituting (29) into (8) yields

$$u(x,t) = \mathrm{e}^{n\,\mathrm{i}x}\Big\{1+\mathrm{i}(m-n^2)t-\frac{1}{2}(m-n^2)^2t^2 - \frac{\mathrm{i}}{6}(m-n^2)^3t^3-\frac{1}{24}(m-n^2)^4t^4 + \frac{\mathrm{i}}{120}(m-n^2)^5t^5-\cdots\Big\}. \quad (30)$$

Consequently, the exact solution of (22),

$$u(x,t) = \mathrm{e}^{\mathrm{i}(nx+(m-n^2)t)}, \quad (31)$$

is readily obtained upon using the Taylor series expansion of $\mathrm{e}^{\mathrm{i}(m-n^2)t}$.

Remark 3. From our results, we can reproduce the exact solutions of (22), in case of $m=2$ or $m=-2$ with $n=1$, that were obtained by Wazwaz [12] from (31) by setting $m=2$ or $m=-2$ with $n=1$.

Example 4. Finally, we consider the cubic nonlinear Schrödinger equation

$$\mathrm{i}u_t+u_{xx}+2|u|^2=0, \quad u(x,0)=2\,\mathrm{sech}(2x). \quad (32)$$

By following the same procedures of example 3, we obtain the system (27) with $m=2$ and the initial conditions can be written as follows:

$$V_i(x,0) = \begin{cases} 2\,\mathrm{sech}(2x), & i=0,\\ 0, & i=1,2,3,\cdots. \end{cases} \quad (33)$$

Solving the system (27) in case of $m=2$, with the initial conditions (33), yields

$$V_0(x,t) = 2\,\mathrm{sech}(2x),$$
$$V_1(x,t) = 8\mathrm{i}t\,\mathrm{sech}(2x),$$
$$V_2(x,t) = -16t^2\,\mathrm{sech}(2x),$$
$$V_3(x,t) = -\frac{64}{3}\mathrm{i}t^3\,\mathrm{sech}(2x), \quad (34)$$
$$V_4(x,t) = \frac{64}{3}t^4\,\mathrm{sech}(2x),$$
$$V_5(x,t) = \frac{256}{15}\mathrm{i}t^5\,\mathrm{sech}(2x).$$

In the same manner, the further components can be simply obtained. Substituting (34) into (8) yields

$$u(x,t) = 2\,\mathrm{sech}(2x)\Big(1+4\mathrm{i}t-8t^2-\frac{32}{3}\mathrm{i}t^3 + \frac{32}{3}t^4+\frac{128}{15}\mathrm{i}t^5-\cdots\Big). \quad (35)$$

The exact solution of (32),

$$u(x,t) = 2\,\mathrm{sech}(2x)\mathrm{e}^{4\mathrm{i}t}, \quad (36)$$

follows immediately upon using the Taylor series expansion of $\mathrm{e}^{4\mathrm{i}t}$.

4. Conclusions

A clear conclusion that can be drawn from our results is that the HPM provides fast convergence to exact solutions. It is also worth noting that the HPM is an effective, simple and quite accurate tool to handle and solve Schrödinger equations and other types of linear and nonlinear problems, having wide applications in engineering, in a unified manner. The two mistakes that happened in [12] have been indicated in this paper. Nonlinear scientific models arise frequently in science and engineering problems to express nonlinear phenomena. The various applications of He's homotopy perturbation method proves that it is an efficient method to handle the nonlinear structure. It is predicted that the HPM will find various applications in science and engineering.

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