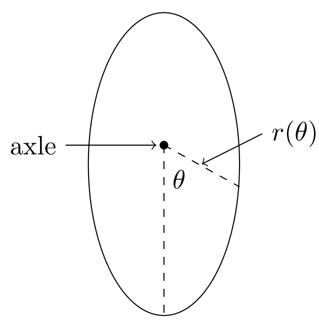
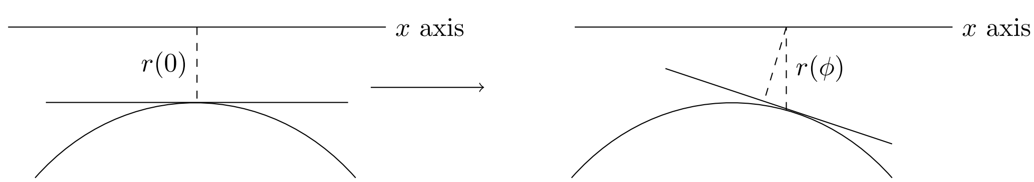
Maple Assignment Part 2
Can a Square Wheel Roll Smoothly Along a Washboard Road?

We solve a more general question: if the radius of a wheel is given by the function $r = r(\theta)$, where θ is measured counterclockwise from the negative y axis (see diagram below), what road, $y = g(x)$ will result in the center of the wheel moving only in the horizontal direction? That is, what road shape will give you a smooth ride?



Finding the Road

We assume the wheel rolls to the right, so that the wheel rolls *clockwise*, and we assume the axle stays on the x axis. As a result, the road has negative y coordinates. The diagram below shows what happens when the wheel is the horizontal line $y = -1$, and the wheel rotates through the angle ϕ.



Notice that the part of the wheel in contact with the road is directly below the axle, and the corresponding radius is $r(\phi)$.

Suppose the road has equation $y = g(x)$. We want to describe the rotation of the wheel in terms of x. Because the axle of the wheel is above the point of contact with the road, x is also the distance travelled by the wheel's axle. The point of contact with the road will be $\big(x, g(x)\big)$ in Cartesian coordinates, and $g(x) = -r(\phi)$. Because there is no slippage, the arclength of the road between 0 and x, and the arclength of the wheel between the angles of 0 and ϕ must be equal:

$$\int_0^x \sqrt{1 + g'(s)^2}\,ds = \int_0^\phi \sqrt{r(\psi)^2 + \big(r'(\psi)\big)^2}\,d\psi$$

We eliminate the integral signs by differentiating with respect to x. On the right-hand side we assume $\phi = \phi(x)$, so the Fundamental Theorem of Calculus gives us

$$\sqrt{1+g'(x)^2} = \sqrt{r(\phi)^2 + \big(r'(\phi)\big)^2}\phi'(x)$$

Because $g(x) = -r(\phi)$, we have $g'(x) = -r'(\phi)\phi'(x)$:

$$\begin{aligned}
\sqrt{1+\big(r'(\phi)\phi'(x)\big)^2} &= \sqrt{r(\phi)^2 + \big(r'(\phi)\big)^2}\phi'(x) \\
1+\big(r'(\phi)\phi'(x)\big)^2 &= \big(r(\phi)^2 + \big(r'(\phi)\big)^2\big)\big(\phi'(x)\big)^2 \\
1 &= \big(r(\phi)\big)^2\big(\phi'(x)\big)^2 \\
\phi'(x) &= \pm\frac{1}{r(\phi)}
\end{aligned}$$

We take the positive root because ϕ increases as x increases, so $\phi'(x)$ is positive. We have

$$\phi'(x) = \frac{1}{r(\phi)}$$

We solve the ODE to get $\phi = \Phi(x)$. The road is now $(x, -r(\Phi(x)))$.

Parametrizing Square Wheels

We assume the square wheel has side length 2, has the axle at the center of the square, and starts with one side parallel to the x axis. Then the bottom side corresponds to $r = \sec\theta$ where $-\pi/4 \le \theta \le \pi/4$. For the right side, we get the same radii, but now associated to angles between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. Similar statements are true for the other two sides. We can therefore describe the radius of the wheel parametrically by composing $\sec\theta$ with the following function:

$$L(\theta) = \begin{cases} & \vdots \\ \theta & \text{if } -\frac{\pi}{4} \le \theta < \frac{\pi}{4} \\ \theta - \frac{\pi}{2} & \text{if } \frac{\pi}{4} \le \theta < \frac{3\pi}{4} \\ \theta - \pi & \text{if } \frac{3\pi}{4} \le \theta < \frac{5\pi}{4} \\ & \vdots \end{cases}$$

We can describe L more succinctly using the sawtooth function $S(x) = x - \lfloor x \rfloor$. If we want to periodically repeat the function $y = x$ over the interval $[a, b]$, we want to transform S so that the transformed sawtooth function that takes the value of a in the bottom of the tooth, and b at the top. We need to increase the period of S to $b - a$, so we consider $S_1(x) = S(\frac{x}{b-a})$. The minimum value has to be moved from 0 to a, so we now have $S_2(x) = S(\frac{x}{b-a}) + a$. The x coordinate of the minimum must also move from 0 to a, so we get $S_3(x) = S_2(x - a) = S(\frac{x-a}{b-a}) + a$. Lastly, the height of the tooth must be $b - a$, so our final function is $T(x) = (b - a)S(\frac{x-a}{b-a}) + a$.

We find that

$$\begin{aligned}
L(x) &= \frac{\pi}{2}\left(\frac{2}{\pi}(x+\frac{\pi}{4}) - \left\lfloor \frac{2}{\pi}(x+\frac{\pi}{4}) \right\rfloor\right) - \frac{\pi}{4} \\
&= x - \frac{\pi}{2}\left\lfloor \frac{2}{\pi}(x+\frac{\pi}{4}) \right\rfloor
\end{aligned}$$

and the wheel will be

$$r(\theta) = \sec\big(T(\theta)\big) = \sec\left(\theta - \frac{\pi}{2}\left\lfloor \frac{2}{\pi}(\theta + \frac{\pi}{4})\right\rfloor\right)$$

Animation of the Wheel Rolling Along the Road

Of course, we want an animation of the wheel rolling along the road. If the center of the wheel is at $(x, 0)$, the points on the edge of the wheel will have been displaced clockwise by $\phi(x)$.

For the following exercises, solve the DE numerically.

EXERCISES
Finding Roads

1. Use Maple to find and plot the road associated to the wheel $r = 1.1 + \sin\phi$.
2. What will $r(\phi)$ be if the wheel is the horizontal line $y = -1$? Find the associated road for $0 \leq \phi < \pi/2 - 0.1$.
3. Find the road associated to $r = 0.9 + \sin\phi$. What fails, and why?

Finding Wheels

4. Use the function above to plot the square wheel (before rotation). Of course, you want to plot in Cartesian coordinates, while the description of the wheel is in terms of $\big(\theta, r(\theta)\big)$.
5. How would you adjust L to produce a regular n-sided wheel? Plot the wheel.
6. Find the roads associated to a square and an equilateral triangle.

Creating Animations

7. Create an animation in Maple of a square wheel rolling along its washboard road smoothly. Note that in each frame you want to draw the entire (rotated) wheel, and, of course, you want to include the road.
8. Try the same with $r = 1.1 + \cos(2\theta)$.