1-9-1999

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Mathematical Sciences Technical Report Series
MSTR 98-06

January 9, 1999

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Abstract

We have all seen many of the beautiful patterns obtained by tiling the hyperbolic plane $\mathbb{H}$ by repeated reflection in the sides of a "kaleidoscopic" polygon. Though there are such patterns on the sphere and the euclidean plane, these positively curved and fiat geometries lack the richness we see in the hyperbolic plane. Many of these patterns have been popularized by the beautiful art of M.C. Escher. For a list of references and a more complete discussion on the construction of artistic tilings see [6].

Contents

1 Kaleidoscopic polygons

We have all seen many of the beautiful patterns obtained by tiling the hyperbolic plane $\mathbb{H}$ by repeated reflection in the sides of a “kaleidoscopic” polygon. Though there are such patterns on the sphere and the Euclidean plane, these positively curved and flat geometries lack the richness we see in the hyperbolic plane. Many of these patterns have been popularized by the beautiful art of M.C. Escher. For a list of references and a more complete discussion on the construction of artistic tilings see [6].

Here is one way to construct a tiling. Select an n-gon $\Delta = P_1 P_2 \cdots P_n$ in the hyperbolic plane such that the measure of its interior angle at the vertex P_i equals π/m_i where m_i is an integer. The polygon Δ is called a *kaleidoscopic polygon* or an $(m_1, m_2, \ldots, m_n)$-polygon. Now create a layer of polygons or tiles surrounding our first polygon by hyperbolically reflecting the first polygon in its edges. Then create a second layer by hyperbolically reflecting in the edges of the second layer and continue on in this manner to construct countably many layers. An example of a tiling constructed from a $(3, 3, 4)$-triangle is given in Figure 1. Let Λ^* be the group generated by the reflections in the sides of a single tile Δ. The Poincaré Polygon Theorem [1, p. 249] ensures that Λ^* is a Fuchsian group, i.e., a discrete group of isometries of $\mathbb{H}$, and Δ is a fundamental region for Λ^*.

A tiling of the plane by $(3, 2, 2, 3)$-quadrilaterals is given in Figure 2. Also note that Figure 1 gives us another example of a quadrilateral tiling. The twelve triangles surrounding the origin make up a kaleidoscopic $(4, 4, 4, 4)$-quadrilateral. Observe that the tiling by $(4, 4, 4, 4)$-quadrilaterals is subdivided by the triangle tiling. We say that the quadrilateral tiling is *divisible*. In fact, the problem of classifying divisible quadrilateral tilings motivated our development of the construction methods given in this paper. See [2] and [3] for more detail on divisible tilings.

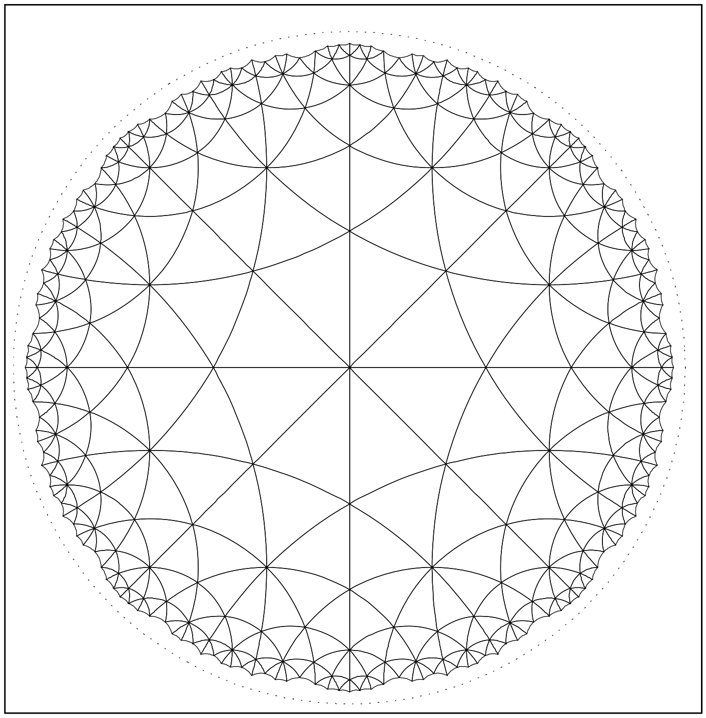

Fig. 1: $(3,3,4)$-triangle tiling of the hyperbolic plane

Let $\Omega \subset \Lambda^*$ be a subgroup of finite index. The primary example is the subgroup Λ (of index 2) consisting of orientation preserving transformations in Λ^*. A fundamental region R for Ω may be constructed from a finite union of tiles. For Λ any two adjacent tiles will do. Now deform this fundamental region into another fundamental region R that is artistically pleasing, and, if desired, paint a pattern on the fundamental region. The Ω-translates of the painted pattern $\bigcup_{g\in\Omega} gR$ form an artistically pleasing pattern that covers the plane without gaps or overlaps. The creation of the images gR, $g \in \Omega$, is easily automated by a computer once a set of generators of Ω, as transformations of $\mathbb{H}$, is known.

Remark 1.1 *For any discrete group of isometries of* $\mathbb{H}$*, a convex polygonal fundamental region exists, which may be constructed from a Dirichlet polygon [1, p. 226]. Thus the program we have been describing can be carried out for any discrete group. For a group generated by reflections in a kaleidoscopic polygon, the Dirichlet polygon must be a tile. For subgroups one may either* *select a collection of tiles or a Dirichlet polygon.*

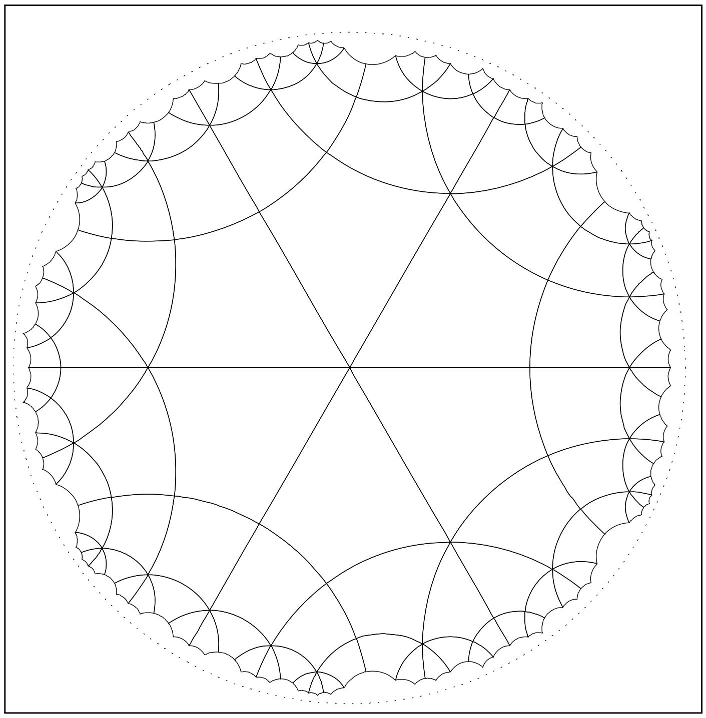

Fig. 2: A (3,2,2,3)-quadrilateral tiling

The group Λ^*, and hence Ω, is easy to work with both algebraically and analytically. From the algebraic point of view, consider the triangle in Figure 1 with a vertex at the origin and whose bottom edge lies on the x-axis. Let p, q, and r denote the reflections in the sides of this triangle. Specifically, p is the reflection across the diameter in the first quadrant, q is the reflection across the x-axis, and r is the hyperbolic reflection in the third circular side. The products pq, qr, and rp are hyperbolic rotations centered at the vertices of the triangle through angles $\pi/4, \pi/3$, and $\pi/3$ respectively, and these rotations have finite orders 4, 3, and 3 respectively. These are the only relations among the reflections, and we obtain a presentation

$$\Lambda^* = \left\langle p, q, r : p^2 = q^2 = r^2 = (pq)^4 = (qr)^3 = (rp)^3 \right\rangle . \qquad (1)$$

Generators for Ω may then be found from p, q, and r; for example $\Lambda = \langle pq, qr, rp \rangle$. From the analytic point of view the formulas for the reflections are easily written down once the sides of the polygon are known. Specifically, suppose that side of a polygon is a portion of the hyperbolic line ℓ defined as the Euclidean circle centered at z_0 and perpendicular to the (dotted) boundary circle. Now, for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $\det A = 1$, let L_A denote the linear fractional transformation defined by $L_A(\overline{z}) = (az + b)/(cz + d)$. Then, the reflection in ℓ is given by

$$z \to L_A(\overline{z}) = \frac{a\overline{z} + b}{c\overline{z} + d}, \text{ where } A = \frac{i}{\sqrt{z_0\overline{z_0} - 1}} \begin{bmatrix} z_0 & -1 \\ 1 & -\overline{z_0} \end{bmatrix}, \qquad (2)$$

If the line is a diameter then z_0 is at infinity and

$$A = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix},$$

where θ is the angle that the diameter makes with the x-axis. The composition of two reflections with matrices A and B is the linear fractional transformation $L_{A\overline{B}}$, in which $\overline{B}$ is obtained by conjugating the entries of B. The generators of Ω are words in p, q, and r as dictated by the geometry of the tiling and the presentation (1). The explicit formulas can then be easily calculated from the basic reflections using the formulas (2). Similar remarks apply to all kaleidoscopic tilings.

Given the situation, it is very useful to find an explicit way to find the sides and vertices of the polygon. Given an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of angles satisfying $0 < \alpha_i \leq \pi/2$, and the hyperbolic condition $\alpha_1 + \alpha_2 + \cdots + \alpha_n < \pi(n-2)$, we would like to find an analytic method for determining all polygons $\Delta = P_1P_2 \cdots P_n$ in the hyperbolic plane such that the interior angle at P_i is α_i. Theorem 7.16.2 of [1] shows how to construct at least one, but it is not enough. In this paper we describe a uniform method to solve this problem for triangles and quadrilaterals; our method is easily extended to higher polygons. The key ingredient in our solution to this problem is the notion of an *angle pencil* or *γ-pencil*, the set of all lines meeting a given line at an angle of measure γ. These angle pencils are parametrized by hyperbolas in the plane, and the polygons may be constructed by finding the intersections of the hyperbolas.

In Section 2, we quickly review some facts from hyperbolic geometry, then briefly discuss convex polygons and some alternate constructions of convex polygons, and then the introduce the space of geodesics. We develop the basic properties of angle pencils in Section 3 and then apply the ideas to the construction of triangles in Section 4 and quadrilaterals in Section 5. The main results are Theorems 4.5 and 5.1, and Propositions 4.3 and 5.2. In Section 6 we pose some homework for the interested reader and some unsolved questions suitable for further research.

For the author, it is a source of satisfaction that most of the conceptual and computational parts of the paper use little more than analytic geometry, calculus, and solving systems of linear and quadratic equations. For the proofs the only boosts required from hyperbolic geometry are angle sum inequalities, and, at one point, the second law of cosines.

The figures in this paper were produced using Maple worksheets and Matlab m-files, which may be found at [7]. Three additional websites that contain material on tilings and geometric constructions are [8], [9], and [10]. The first gives a Java script enabled visualization of the genus 2 Teichmüller space by showing octagons that generate a Fucshian group corresponding to a genus 2 surface. The second allows basic hyperbolic geometric constructions and reflections, i.e., a Java script geometric sketchpad. The third site has downloadable software that is a hyperbolic geometric sketchpad.

2 Hyperbolic geometry

Our basic reference for hyperbolic geometry is [1]. We use the disc model for the hyperbolic plane $\mathbb{H}$, in which the points are in the interior of the unit disc in the complex plane, the lines are the unit disc portions of circles and lines perpendicular to the boundary of the unit disc, and reflections are inversions in the circles defining the lines. In the figures in this paper the boundary of the unit disc is always drawn with a dotted circle. We recall a few facts about distance, angles, and lines in the hyperbolic geometry of the disc that we shall use frequently.

1. For $z, w \in \mathbb{H}$ the *hyperbolic distance* from z to w is denoted by $\rho(z, w)$. All we need to know is that for $z \in \mathbb{H}$,

$$\rho(z, 0) = 2 \tanh^{-1}(|z|). \tag{3}$$

   Alternatively, if a point is at a hyperbolic distance h from the origin then it lies on the circle with Euclidean radius

$$r = \tanh(h/2), \tag{4}$$

   centred at the origin.

2. For intersecting lines, the *angle* between the lines at the point of intersection is the Euclidean angle between the curves. Remark 3.5 discusses how to compute this angle.

3. The sum of the interior angles in an n-gon satisfies

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n < \pi(n-2).$$

This includes the case where vertices are on the boundary and the corresponding angles have measure 0.

4. The second law of cosines states that if the measures of the angles of a non-degenerate triangle are α, β, and γ and the side opposite the angle with measure γ has length c, then

$$\cosh c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}. \tag{5}$$

The vertex corresponding to γ may be on the boundary, in which case $\gamma = 0$.

5. A non-diameter hyperbolic line is the intersection with $\mathbb{H}$ and a circle of radius r and center $z_0 = a + bi$, meeting the unit circle at a right angle. The equation of the hyperbolic line is given by $|z - z_0|^2 = r^2$. Since the circle is orthogonal to the unit circle,

$$|z_0|^2 = a^2 + b^2 = 1 + r^2. \tag{6}$$

Therefore, the circle equation may be rewritten as

$$2ax + 2by = 1 + x^2 + y^2. \tag{7}$$

We call the point z_0 and radius r the *Euclidean center* and *Euclidean radius* of the line, respectively.

6. Rewrite $2ax + 2by = 1 + x^2 + y^2$ as

$$\frac{a}{r}x + \frac{b}{r}y = \frac{1 + x^2 + y^2}{2r}. \tag{8}$$

If we let $z_0 \to \infty$, the limiting equation gives us a diameter line

$$a_\infty x + b_\infty y = 0, \tag{9}$$

where $a_\infty = \lim_{z_0\to\infty} a/r$ and $b_\infty = \lim_{z_0\to\infty} b/r$. Note that

$$a_\infty^2 + b_\infty^2 = 1. \tag{10}$$

7. The endpoints of the line determined by (7) lie on the unit circle $x^2 + y^2 = 1$. By replacing $x^2 + y^2$ by 1 in (7) we get equations for the end points:
$$x^2 + y^2 = 1,\ ax + by = 1 \tag{11}$$
in the generic case and
$$x^2 + y^2 = 1,\ a_\infty x + b_\infty y = 0 \tag{12}$$
in the diameter case.

8. Following the terminology of [1] we call two lines that meet in the interior of the hyperbolic plane *intersecting lines,* two lines that meet on the boundary of $\mathbb{H}$ *parallel lines,* and two lines that have no intersection at all *disjoint lines.*

Convex polygons. A convex hyperbolic polygon with interior angles (α_1, α_2, ..., α_n) exists if and only $\alpha_1 + \alpha_2 + \cdots + \alpha_n < \pi(n-2)$, and $\alpha_i < \pi$, [1, Theorem 7.16.2]. We are interested only in the case in which all angles are acute or right angles, i.e., $\alpha_i \leq \pi/2$, since we are really interested in integer submultiples of π. Though we have limited the scope of our work to convex polygons with acute or right angles, the methods could also be applied to determining all convex polygons.

The set of congruence classes of convex polygons with prescribed angles has several degrees of freedom. Using the implicit function theorem and the fact that there is at least one such polygon, it is not hard to show that the number of degrees of freedom is at least $n - 3$. However, we discuss the exact structure of this set only for $n \leq 4$. The AAA congruence theorem for hyperbolic geometry shows that there is a single congruence class for triangles. For convex quadrilaterals with acute or right angles it follows from our theorem that the set of congruence classes corresponds to a semi-infinite open interval.

It would be interesting and convenient to be able to build our quadrilaterals from simply constructed subpolygons. In [1, Chapter 7], Beardon illustrates some of these constructions as he discusses triangles, quadrilaterals, pentagons, and hexagons. In particular, he shows in the proof of his Theorem 7.16.2, how one may build a convex polygon with prescribed angles $\alpha_1, \ldots, \alpha_n$ out of right angled triangles. Unfortunately, his method produces only a single congruence class of such polygons, whereas the set of congruence classes of convex polygons with prescribed angles can be shown to have at least $n-3$ degrees of freedom. Beardon's method allows us to construct all triangles, but not all quadrilaterals.

We might also construct the polygons by putting together four Lambert quadrilaterals. A Lambert quadrilateral has three right angles and a fourth acute angle, say ϕ. There are two degrees of freedom in prescribing the sidelengths of Lambert quadrilaterals and there are nice trigonometric formulas relating the sides [1, p. 157]. If we take four of these quadrilaterals (with sidelengths chosen appropriately) and join all four together at the right angle opposite the acute angle then we get a quadrilateral with angles $\phi_1, \ldots, \phi_4$. It turns out that with the angles fixed, a one parameter the family of quadrilaterals can be constructed. However, by repeated use of the formulas [1, Theorem 7.17.1] we discover that the angles must satisfy the equation $\cos\phi_1 \cos\phi_3 = \cos\phi_2 \cos\phi_4$. Because of this constraint, not all required quadrilaterals can be obtained by putting together four Lambert quadrilaterals.

The space of geodesics. We have parametrized the set of non-diameter lines by their Euclidean centers and have separately parametrized the diameter lines by the unit circle (antipodal points identified). Though these parametrizations are best for computations and for our results on angle pencils, a simultaneous parametrization for all lines would be useful, especially when dealing with continuity issues. We use (8) to construct one. For $z_0 \in U = \{z \in \mathbb{C} : |z| > 1\}$ define

$$w_0 = \frac{z_0}{r} = \frac{z_0}{\sqrt{|z_0|^2 - 1}}.$$

Note that

$$z_0 = \frac{w_0}{\sqrt{|w_0|^2 - 1}} \text{ and } r = \frac{1}{\sqrt{|w_0|^2 - 1}}.$$

Using the parameter $w_0 = c + di \in \overline{U} = \{z \in \mathbb{C} : |z| \geq 1\}$ and the function $s = \left(|w_0|^2 - 1\right)^{1/2}$ we may rewrite (8) as

$$cx + dy = \frac{s}{2}(1 + x^2 + y^2). \tag{13}$$

Since $s = 0$ for a diameter line, (13) gives a continuous parametrization of all lines at once, but it is not as convenient as our standard parametrization. If we identify antipodal points on the circle, the resulting space $\mathcal{U} = \overline{U} / \backsim$ is a line bundle over the circle giving us a one-to-one parametrization of the lines in $\mathbb{H}$ by a manifold. In particular, if we have a 1-parameter family of lines in $\mathbb{H}$ then a curve on $\mathcal{U}$ is determined.

Though we do not make use of the fact in this paper, it is worth noting that $\mathcal{U}$ is homeomorphic to $\mathcal{M}$, the Moebius band without boundary, pictured in Figure 3. The curves and lines pictured on $\mathcal{M}$ are called *latitudes* and *meridians*. We may think of $\mathcal{M}$ as being constructed from a short, wide, Mercator map twisted and pasted together along the international date line. Latitude lines on the map become the curves on $\mathcal{M}$, and meridians (longitude lines) become the straight lines. To show the homeomorphism let $V = \mathbb{R} \times (-1, 1)$ and $W = \mathbb{R} \times [0, 1) \subset V$. There is a map $p : W \longrightarrow \overline{U}$ defined by

$$p : (\theta, t) \longrightarrow \frac{e^{i\theta}}{1-t} = re^{i\theta}, \tag{14}$$

and a map $q : V \longrightarrow \mathcal{M} \subset \mathbb{R}^3$ given by

$$\begin{aligned} q : (\theta, t) \longrightarrow & A(\cos(2\theta)\overrightarrow{\mathbf{i}} + \sin(2\theta)\overrightarrow{\mathbf{j}}) \\ & + Bt\left(\sin\theta(\cos(2\theta)\overrightarrow{\mathbf{i}} + \sin(2\theta)\overrightarrow{\mathbf{j}}) + \cos\theta\overrightarrow{\mathbf{k}}\right), \end{aligned} \tag{15}$$

where A and B are constants selected to eliminate self-intersections in $\mathbb{R}^3$. Both p and q are universal covering space projections, and have the covering transformation properties $p(\theta + \pi, t) = p(\theta, t)$, and $q(\theta + \pi, t) = q(\theta, -t)$. Even though $p^{-1}(re^{i\theta}) = (\theta, (r-1)/r)$ is an ambiguous multi-valued map, the composition $q \circ p^{-1} : \overline{U} \longrightarrow \mathcal{M}$ is a well-defined continuous map and a homeomorphism when restricted to U. Since $q(\theta + \pi, 0) = q(\theta, 0)$, it follows easily that the induced quotient map $q \circ p^{-1} : \mathcal{U} = \overline{U}/ \backsim \longrightarrow \mathcal{M}$ is a homeomorphism. When we set $t = 0$ we trace out the central latitude on the Moebius band and the quotient boundary $(\partial\overline{U})/ \backsim$ of the boundary of $\overline{U}$; these points correspond to the set of diameter lines, i.e., the elliptic pencil of lines that pass through the origin. A non-zero value of t corresponds to all lines that have a given Euclidean radius, i.e., the pencil of lines tangent to a hyperbolic circle centered at the origin. The corresponding latitude seems to wrap around the band twice. Meridians on the band correspond to the hyperbolic lines, whose centres all lie on a diameter line through the origin, but are exterior to the unit disc, i.e., the hyperbolic pencil of lines perpendicular to a diameter line. The points on the boundary of the band correspond to points on the boundary of the hyperbolic disc.

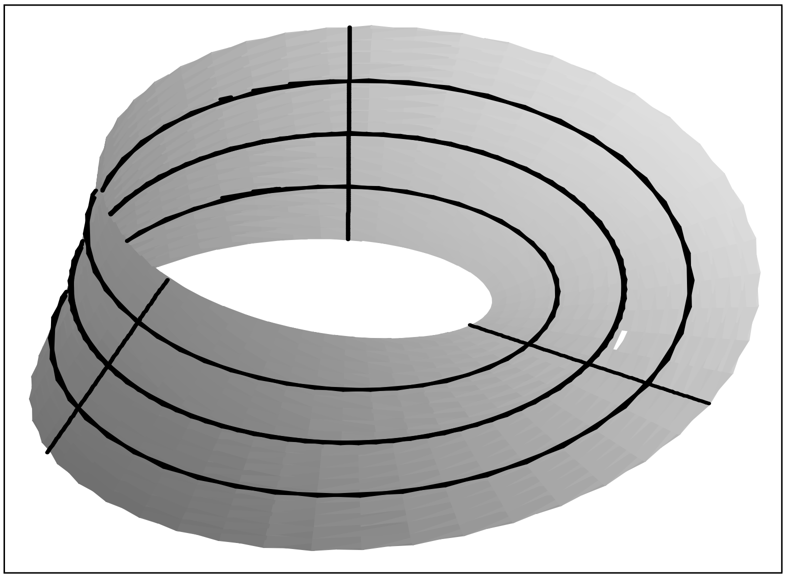

Fig. 3: The space of geodesics $\mathcal{U}$ as a Moebious band

3 Angle pencils

Let ℓ and ℓ' be two lines meeting at the point w in the unit disc. The two counter-clockwise angles from ℓ to ℓ' at w are equal to each other because of the opposite angle theorem. Thus we may speak unambiguously of the measure of the counter-clockwise angle from ℓ to ℓ'. A clockwise angle is defined similarly. For two lines tangent at the boundary, clockwise and counter-clockwise angles still make sense, though they must be 0 or π.

Definition 3.1 *For* $0 < \gamma \leq \pi/2$ *the* counter-clockwise angle pencil *or* γ*-pencil with base* ℓ', $\mathcal{P}^+(\ell', \gamma)$, *is the set of all lines* ℓ *forming a counter-clockwise angle from* ℓ *to* ℓ' *of measure* γ. *The* clockwise angle pencil *or* γ*-pencil,* $\mathcal{P}^-(\ell', \gamma)$, *is defined analogously.*

A picture of some of the lines of the pencil $\mathcal{P}^+(\ell', \pi/4)$ is given in Figure 11, when ℓ' is a line nearly equal to the x-axis. If $\gamma = \pi/2$ we get the standard notion of a hyperbolic pencil [1, p. 170] and the clockwise and counter-clockwise pencils are the same.

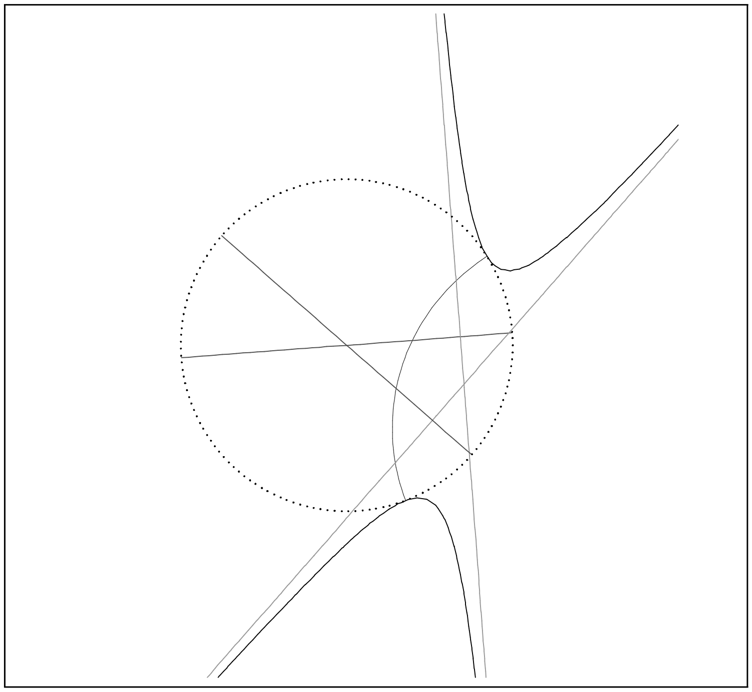

Fig. 4: Locus of Euclidean centers of lines cutting at 45°

Now let ℓ' be a fixed non-diameter line, let $\ell \in \mathcal{P}^+(\ell', \gamma)$ be arbitrary, and let $A+Bi, R$, and $a+bi$, r be the Euclidean centers and radii of ℓ' and ℓ respectively. The values A, B, a, and b satisfy the *pencil equation*

$$(Aa + Bb - 1)^2 = (A^2 + B^2 - 1)(a^2 + b^2 - 1)\cos^2\gamma, \tag{16}$$

as we show in Proposition 3.3. Now replace $A^2 + B^2 - 1$ by R^2 and divide both sides of (16) to get

$$(\frac{A}{R}a + \frac{B}{R}b - \frac{1}{R})^2 = (a^2 + b^2 - 1)\cos^2\gamma. \tag{17}$$

Pick a sequence of hyperbolic lines converging to a diameter line ℓ' with equation $A_\infty x + B_\infty y = 0$. Then (17) becomes

$$(A_\infty a + B_\infty b)^2 = (a^2 + b^2 - 1)\cos^2\gamma. \tag{18}$$

The same equations hold for $\ell \in \mathcal{P}^-(\ell', \gamma)$.

Remark 3.2 *Analogous equations may be derived for other types of pencils. Some examples are posed as homework in Section 6.*

Either of the equations (16) or (18) defines a conic section in the plane (in fact a hyperbola), which we denote by $H(\ell', \gamma)$. To show the relation of the hyperbola to other geometric objects associated to the angle pencil we have drawn the following in Figure 4: the dotted boundary circle, the base line of the angle pencil ℓ' with Euclidean centre $A + Bi = \frac{3}{2} - \frac{1}{2}i$, the hyperbola for $\gamma = \pi/3$, and the asymptotes of the hyperbola. The two *points at infinity* on the hyperbola correspond to the two diameter lines in the angle pencil that cross ℓ' at angle γ. We have also drawn these lines in Figure 4, where we note that $H(\ell', \gamma)$ is tangent to the boundary circle at the endpoints of ℓ'. Each asymptote in Figure 4 meets, at a right angle, the diameter line in one of angle pencils.

Let us formalize the previous discussion into two propositions.

Proposition 3.3 *Let ℓ' be a hyperbolic line, let $0 < \gamma \leq \pi/2$, and let $\mathcal{P}^+(\ell', \gamma)$ and $\mathcal{P}^-(\ell', \gamma)$ be their angle pencils. Then*

i) The hyperbolic plane $\mathbb{H}$ is a disjoint union of the lines in $\mathcal{P}^+(\ell', \gamma)$. In fact, any two distinct ℓ_1, $\ell_2 \in \mathcal{P}^+(\ell', \gamma)$ are disjoint lines. The map $\ell \to \ell \cap \ell'$ is a one-to-one correspondence between $\mathcal{P}^+(\ell', \gamma)$ and the points of the ℓ'. Similar statements hold for $\mathcal{P}^-(\ell', \gamma)$.

ii) If ℓ' has Euclidean center $A + Bi$ and $\ell \in \mathcal{P}^+(\ell', \gamma) \cup \mathcal{P}^-(\ell', \gamma)$ has Euclidean center $a + bi$, then (16) holds. Thus, the hyperbola defined by (16) (and the points at infinity) corresponds to the union $\mathcal{P}^+(\ell', \gamma) \cup \mathcal{P}^-(\ell', \gamma)$ of both the clockwise and counter-clockwise pencils.

iii) If ℓ' is a diameter with equation $A_\infty x + B_\infty y = 0$, where $A_\infty^2 + B_\infty^2 = 1$, and if $\ell \in \mathcal{P}^+(\ell', \gamma) \cup \mathcal{P}^-(\ell', \gamma)$ has Euclidean center $a + bi$, then (18) holds.

iv) For $x \in \ell'$ let $\ell^+(x)$ be the line ℓ in $\mathcal{P}^+(\ell', \gamma)$ such that $\ell \cap \ell' = \{x\}$. Then $x \to \ell^+(x)$ maps ℓ' to a curve $\mathcal{L}$ in $\mathcal{U}$, diffeomorphic to $\mathbb{R}$.

Proposition 3.4 *Let $H(\ell', \gamma)$ be the conic section defined by either (16) or (18). First assume that $0 < \gamma \leq \pi/2$. Then the following hold.*

i) The conic section $H(\ell', \gamma)$ is a hyperbola tangent to the unit circle at the endpoints of ℓ'.

ii) Each of $\mathcal{P}^+(\ell', \gamma)$ and $\mathcal{P}^-(\ell', \gamma)$ has one diameter line, and these two diameter lines correspond to the points at infinity of $H(\ell', \gamma)$.

iii) The asymptotes of $H(\ell', \gamma)$ are perpendicular to the diameters in the pencils $\mathcal{P}^+(\ell', \gamma)$ and $\mathcal{P}^-(\ell', \gamma)$.

If $\gamma = \pi/2$, then

iv) $H(\ell', \gamma)$ is a degenerate hyperbola. It consists of the points that lie on the Euclidean line passing through the endpoints of ℓ', but are exterior to the unit disc.

Proof of Proposition 3.3. Let x be any point not on ℓ' and suppose $y \in \ell'$. As y travels from one end of ℓ' to the other, the counter-clockwise angle between the geodesic $\overleftrightarrow{xy}$ and ℓ' varies from 0 to π or vice versa. Thus x lies on at least one line in the pencil. Suppose that two distinct lines in the pencil are not disjoint. Then we have a triangle with interior angles γ and $\pi - \gamma$, and hence the sum of the angles is greater than π, a contradiction. The rest of *i*) is straightforward.

Previous discussion showed how (18) follows from (16), hence *iii*) follows from *ii*). In turn, (16) may be derived simply from the Euclidean law of cosines. As shown in Figure 5, draw the unit circle and the two circles corresponding to the line ℓ' with Euclidean center $z_0 = A + Bi$ and a line ℓ with Euclidean center $z_1 = a + bi$. Let R and r be their respective Euclidean radii. Let z be the point of intersection of the two lines. The Euclidean triangle $\Delta z_0 z_1 z$ has either angle γ or angle $\pi - \gamma$ at z since the hyperbolic lines ℓ and ℓ' meet at angle γ. Thus $\cos(\angle z_0 z z_1) = \pm \cos\gamma$. Apply the law of cosines to get

$$|z_0 - z_1|^2 = |z_0 - z|^2 + |z_1 - z|^2 - 2\,|z_0 - z|\,|z_1 - z| \cos(\angle z_0 z z_1)$$

or

$$(A - a)^2 + (B - b)^2 = R^2 + r^2 \pm 2Rr\cos\gamma.$$

Substitute $R^2 = (A^2 + B^2 - 1)$ and $r^2 = (a^2 + b^2 - 1)$, expand, and simplify to get $1 - Aa - Bb = \pm Rr\cos\gamma$; square both sides and make the substitutions again to arrive at (16). The proof of *iv*) is not difficult and is left to the reader. ■

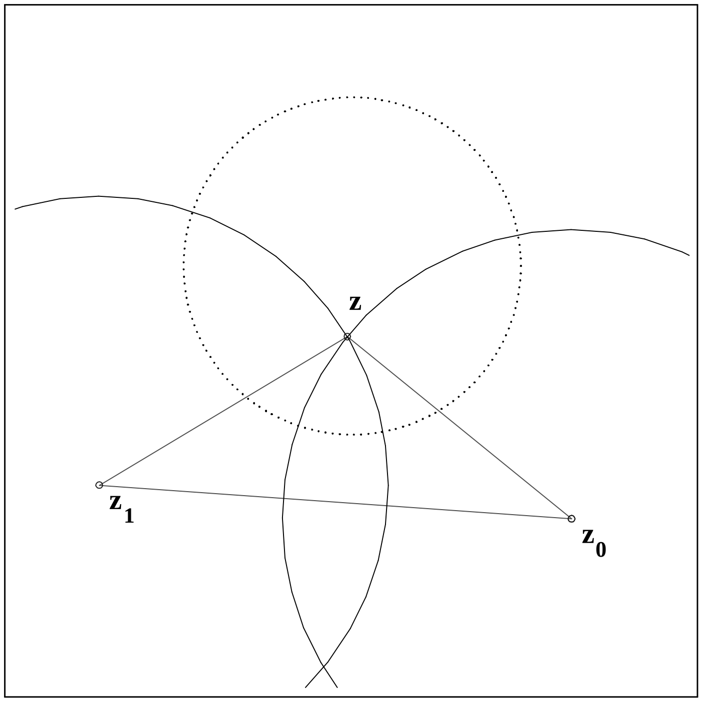

Fig. 5: Law of cosines configuration

Remark 3.5 *The proof of Proposition 3.3 also shows that either angle between ℓ and ℓ' satisfies*

$$\cos^2\gamma = \frac{(Aa+Bb-1)^2}{(A^2+B^2-1)(a^2+b^2-1)}. \tag{19}$$

Let $s = (a^2+b^2-1)^{-1/2}$, $S = (A^2+Ab^2-1)^{-1/2}$, $c+di = s(a+bi)$, and $C+Di = S(A+Bi)$. Then, (19) can be converted to an equation uniformly valid on $\overline{U}$:

$$\cos\gamma = |Cc+Dd-Ss|\,. \tag{20}$$

Remark 3.6 *Unfortunately, the determination of $\cos\gamma$ does not tell us how to compute clockwise and counter-clockwise angles. We may do this in two steps. First compute z, the intersection point of the two lines. There are two solutions or no solutions to the system of circle equations defining the line. The one we seek is the only one inside the unit circle. Now the tangent line directions to the hyperbolic lines are $\pm i(z-z_0)$ and $\pm i(z-z_1)$. If we* *write these numbers in polar form we can easily compute the clockwise and counter-clockwise angles.*

Proof of Proposition 3.4. The conic section $H(\ell', \gamma)$ is a hyperbola if the discriminant

$$D = \frac{\partial^2 h}{\partial a^2}\frac{\partial^2 h}{\partial a^2} - \left(\frac{\partial^2 h}{\partial a \partial b}\right)^2$$

is negative, where $h(a, b) = (Aa + Bb - 1)^2 - (A^2 + B^2 - 1)(a^2 + b^2 - 1)\cos^2\gamma$ in the generic case or $h(a, b) = (A_\infty a + B_\infty b)^2 - (a^2 + b^2 - 1)\cos^2\gamma$ in the case of a diameter. After differentiating, expanding, factoring, and using the trigonometric substitution $\cos^2\gamma = 1 - \sin^2\gamma$ we get

$$D = -4(\cos^2\gamma)(A^2 + B^2 - 1)((A^2 + B^2)\sin^2\gamma + 1),$$

in the generic case or

$$D = -4(\cos^2\gamma)(\sin^2\gamma),$$

in the diameter case. These are both negative.

From (16) or (18) we see that $a^2 + b^2 - 1$ is non-negative and hence $H(\ell', \gamma)$ does not enter the interior of the unit disc. If (a, b) lies on the unit disc and the hyperbola, then, from (16) or (18), we see that $Aa + Bb = 1$ or that $A_\infty a + B_\infty b = 0$. It follows from (11) or (12) that (a, b) is an endpoint of ℓ'. Also, it follows from the geometric discussion that the unit circle and $H(\ell', \gamma)$ are tangent where they meet. Alternatively, we can verify tangency algebraically by checking collinearity of the gradients of the defining equations of $H(\ell', \gamma)$ and the unit circle, at the endpoints of ℓ'. The gradients are collinear if and only if the quantity $a\partial h/\partial b - b\partial h/\partial a = 0$. Now $a\partial h/\partial b - b\partial h/\partial a$ simplifies to $4(Ab - Ba)(Aa + Bb - 1)$ or $4(A_\infty b - B_\infty a)(A_\infty a + B_\infty b)$. These are zero at the endpoints of ℓ', by equations (11) or (12). This proves *i*).

To prove *ii*) consider a variable diameter ℓ. As the diameter ℓ moves through all positions the angle of the clockwise intersection with ℓ' varies from 0 to π or vice versa. By continuity there is at least one point where the intersection angle has measure γ. There cannot be more than one because all the members of the pencil $\mathcal{P}^+(\ell', \gamma)$ are disjoint. Now let $a + bi$ be a variable point on the hyperbola. Let $a + bi$ go to infinity along one of the four branches of the hyperbola and let

$$c + di = \lim_{a+bi\to\infty} \frac{a + bi}{\sqrt{a^2 + b^2 - 1}}.$$

By continuity in $\mathcal{U}$ the limiting line has equation $cx + dy = 0$ according to (13). Also by continuity the angle of intersection is γ. Thus this limit corresponds to the unique diameter in the pencil. Since $c + di$ is one of the unit directions of the corresponding asymptote of the hyperbola, the equation $cx + dy = 0$ immediately implies that the asymptote is perpendicular to the corresponding diameter. Note that the two points of intersection of the hyperbola with the unit circle splits the hyperbola into four parts, two of which are associated with each asymptote. The points of the hyperbola corresponding to a given pencil, (either $\mathcal{P}^+(\ell', \gamma)$ or $\mathcal{P}^-(\ell', \gamma)$), are the following: the two parts whose asymptote meets the diameter line of the pencil at right angles, and the point at infinity where the two parts meet the asymptote.

When $\gamma = \pi/2$ the hyperbola must be degenerate since the discriminant is zero. Equation (16) becomes $Aa + Bb - 1 = 0$. However this is just the equation of the line determining the endpoints of ℓ' (see (11) or (12)). ∎

Remark 3.7 *Determining the curves traced out in $\mathcal{U}$ by angle pencils may add some geometric insight, even though it may not help us computationally. See Section 6 for a problem statement.*

4 Triangle construction

We are now ready to construct triangles with prescribed angles α, β, and γ satisfying $\alpha + \beta + \gamma < \pi$ and $\alpha, \beta, \gamma \leq \pi/2$. There are two possible methods of construction, one by direct geometric methods using the second law of cosines and the other using angle pencils. The first method is more direct but the pencil method gives more information. By using a hyperbolic isometry we may assume that our triangle is congruent to the candidate triangle ΔABC in Figure 6. We assume that the vertex with angle α is at the origin A, that $\ell' = \overleftrightarrow{AB}$ is the x-axis and that the line $\ell'' = \overleftrightarrow{AC}$ forms a second side lying in the first quadrant. We want to place B and C so that $\ell = \overleftrightarrow{BC}$ that meets ℓ' in a counter-clockwise angle of measure β and meets ℓ'' in a clockwise angle of measure γ, with both intersections in the first quadrant.

Direct Method. The second law of cosines says that the lengths of segments $\overline{AB}$ and $\overline{AC}$ are given by

$$\cosh(AB) = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta},$$
$$\cosh(AC) = \frac{\cos\alpha\cos\gamma + \cos\beta}{\sin\alpha\sin\gamma}.$$

From (4) we see that the Euclidean distances from the origin are given by

$$r_B = \tanh\left(\frac{1}{2}\cosh^{-1}\left(\frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}\right)\right), \tag{21}$$
$$r_C = \tanh\left(\frac{1}{2}\cosh^{-1}\left(\frac{\cos\alpha\cos\gamma + \cos\beta}{\sin\alpha\sin\gamma}\right)\right).$$

Thus $B = r_B$, $C = r_C e^{i\alpha}$. According to (7), the Euclidean center $z_0 = a + bi$ of $\overleftrightarrow{BC}$ is found by solving the two linear equations

$$2ar_B = 1 + r_B^2,\ 2ar_C\cos\alpha + 2br_C\sin\alpha = 1 + r_C^2. \tag{22}$$

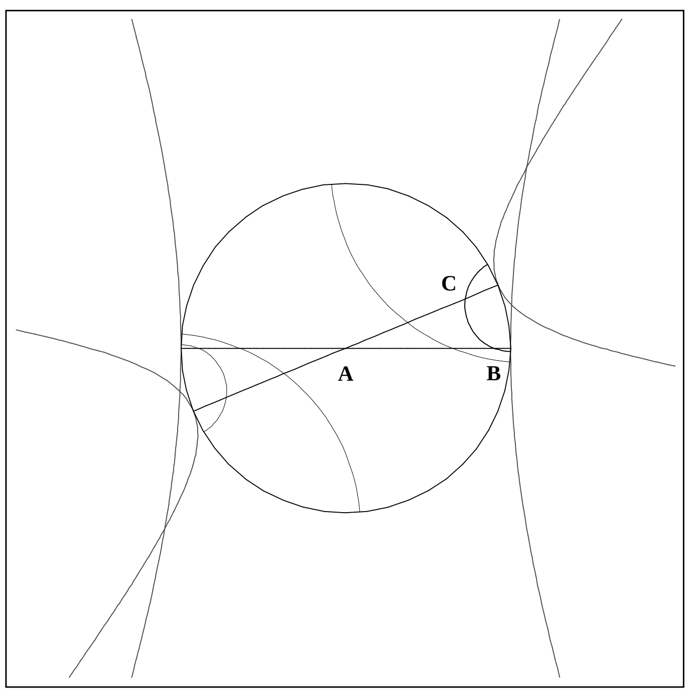

Fig. 6: Standard triangle and pencil hyperbolas

Remark 4.1 *Though it is not immediately obvious, the centre* z_0 *depends algebraically on the trigonometric values of* α, β, *and* γ. *We have seen that the group* $\Lambda = \langle pq, qr, rp\rangle$ *is generated by three linear fractional transformations*

$$pq \longrightarrow \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \overline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$$

$$qr \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overline{\left(\frac{i}{\sqrt{z_0\overline{z_0}-1}} \begin{bmatrix} z_0 & -1 \\ 1 & -\overline{z_0} \end{bmatrix} \right)} = \frac{-i}{\sqrt{z_0\overline{z_0}-1}} \begin{bmatrix} \overline{z_0} & -1 \\ 1 & -z_0 \end{bmatrix}$$

$$rp \longrightarrow \left(\frac{i}{\sqrt{z_0\overline{z_0}-1}} \begin{bmatrix} z_0 & -1 \\ 1 & -\overline{z_0} \end{bmatrix} \right) \overline{\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}} = \frac{i}{\sqrt{z_0\overline{z_0}-1}} \begin{bmatrix} z_0 e^{-i\alpha} & -e^{i\alpha} \\ e^{-i\alpha} & -\overline{z_0}e^{i\alpha} \end{bmatrix}.$$

It then follows that all entries are algebraic numbers and hence that $\Lambda \subset PSL_2(K)$ *for some number field* K *that contains* i.

Remark 4.2 *Even though we have solved the triangle construction problems very simply, we explore the pencil method for the following reasons:*

- *It is the first step of a uniform method to construct all convex polygons.*
- *The solution requires an analysis of the common lines of two angle pencils with intersecting base lines. The case for disjoint base lines is required for the case of convex quadrilaterals with acute angles. The combination of these two cases is used to determine all higher polygons and convex polygons with obtuse angles.*
- *Finding* z_0 *exactly as an algebraic number is simpler, since one solves the equations (23). The algebraic relations can also be found from (21) and (22), though the method does not extend to quadrilaterals.*

Pencil Method In our standard configuration the line ℓ lies in $\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)$, and, in turn, ℓ corresponds to one of the four possible intersection points of $H(\ell', \beta) \cap H(\ell'', \gamma)$. The following proposition formalizes the relation between intersections of angle pencils and the intersections of the hyperbolas. We defer the proof of the proposition to the end of the section.

Proposition 4.3 *Let* ℓ' *and* ℓ'' *be two intersecting lines in the hyperbolic plane. Assume that the lines have been labelled so that counter-clockwise angle* α *from* ℓ' *to* ℓ'' *satisfies* $\alpha \le \pi/2$. *Also assume that* β *and* γ *satisfy* $0 < \beta, \gamma \le \pi/2$, *and* $\alpha + \beta + \gamma < \pi$. *Then under various conditions on* α, β, *and* γ *certain of the angle pencil intersections* $\mathcal{P}^+(\ell', \beta) \cap$

$\mathcal{P}^+(\ell'', \gamma), \ldots, \mathcal{P}^-(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)$ are non-empty. The number of lines in the non-empty intersections and the corresponding conditions on α, β, and γ are shown in Table 1. The intersections not shown contain only redundancies arising from identifications $\mathcal{P}^+(\ell', \pi/2) = \mathcal{P}^-(\ell', \pi/2)$. In particular:

i) If an intersection is non-empty then it contains two lines.

ii) The intersection $\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)$ always contains exactly two lines.

iii) The Euclidean centers of the lines in the intersections correspond to points in the intersection $H(\ell', \beta) \cap H(\ell'', \gamma))$ (diameter lines in $\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)$ correspond to intersection at infinity in $H(\ell', \beta) \cap H(\ell'', \gamma)$).

Case	Conditions and intersections
i)	$\alpha \le \beta + \gamma,\ \beta \le \alpha + \gamma,\ \gamma \le \alpha + \beta,\ \lvert\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)\rvert = 2$
ii)	$\alpha + \beta < \gamma,\ \lvert\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^+(\ell'', \gamma)\rvert = \lvert\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)\rvert = 2$
iii)	$\alpha + \gamma < \beta,\ \lvert\mathcal{P}^-(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)\rvert = \lvert\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)\rvert = 2$
iv)	$\beta + \gamma < \alpha,\ \lvert\mathcal{P}^-(\ell', \beta) \cap \mathcal{P}^+(\ell'', \gamma)\rvert = \lvert\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)\rvert = 2$

Table 1. Pencil Intersections

Figures 7, 8, 9, and 10 illustrate the four cases in Proposition 4.3. In Figure 7, $\alpha = \pi/3$, $\beta = \pi/4$, and $\gamma = \pi/5$; in Figure 8, $\alpha = \pi/7$, $\beta = \pi/12$, and $\gamma = \pi/3$. Figure 9 is the same as Figure 8 with β and γ reversed, and in Figure 10, $\alpha = \pi/3$, $\beta = \pi/6$, and $\gamma = \pi/12$. The lines ℓ' and ℓ'' are the lines connecting the two points of tangency of the hyperbolas. In Figure 7 it appears that there may be two additional solutions to the equations outside the plotting window. However if the window is enlarged the hyperbolas are observed to curve away from each other instead of meeting. Unfortunately making the window large enough to see this renders the detail in the unit circle too small for a good view. Furthermore, in constructing the pencil intersections, two real and two complex points were found as the intersection of the two hyperbolas, confirming that there are two points of intersection. Analogous remarks apply to Figure 10 except that there are four real solutions and one intersection outside the viewing window. In Figures 8 and 9 the four intersection points are all visible.

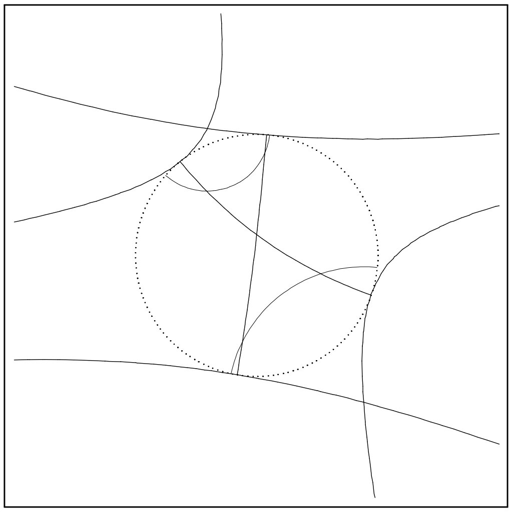

Fig 7: $\alpha \leq \beta+\gamma, \beta \leq \alpha+\gamma, \gamma \leq \alpha+\beta$

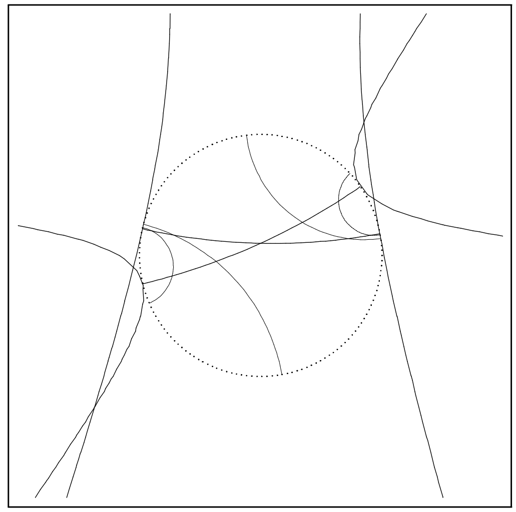

Fig. 8: $\alpha+\beta<\gamma$

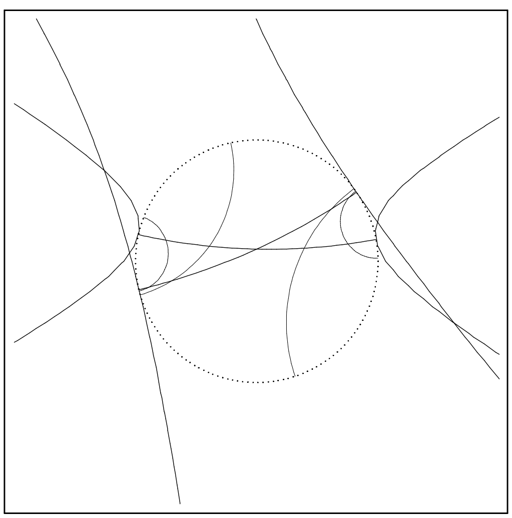

Fig. 9: $\alpha+\gamma<\beta$

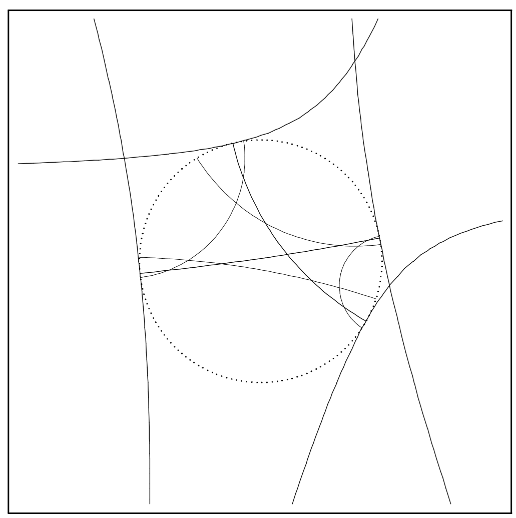

Fig. 10: $\beta+\gamma<\alpha$

Remark 4.4 *The fact that each non-empty intersection of pencils has two elements has a simple geometric explanation. A* 180° *hyperbolic rotation about the point of intersection of* ℓ' *and* ℓ'' *interchanges the lines in the pencil intersections.*

Proposition 4.3 allow us to complete the discussion of our standard first quadrant construction of a triangle, illustrated in Figure 6. According to the proposition there is exactly one line of the desired type whose intersections with ℓ' and ℓ'' lie in the first quadrant. The Euclidean center of this line must lie in the first quadrant outside the unit circle. In our pencil equations (18) we may take $A_\infty + B_\infty i = i$ for ℓ' and $A_\infty + B_\infty i = -\sin\alpha + i\cos\alpha$ for ℓ''. Our two pencil equations become

$$\begin{aligned}(a^2+b^2-1)\cos^2\beta &= b^2,\\ (a^2+b^2-1)\cos^2\gamma &= (-a\sin\alpha + b\cos\alpha)^2.\end{aligned} \tag{23}$$

These equations are easily solved numerically for a and b by a standard package such as Maple. It is also possible to solve the equations by elimination using at most the quadratic formula and the extraction of square roots. Let us formalize the construction in the following theorem.

Theorem 4.5 *Let $\alpha, \beta,$ and γ satisfy $\alpha + \beta + \gamma < \pi$ and $\alpha, \beta, \gamma \le \pi/2$. Then there is a unique triangle ABC such that*

$$\angle CAB = \alpha,\ \angle ABC = \beta,\ \angle BCA = \gamma,$$

A is the origin, B lies on the real axis, and C lies on first quadrant radial line meeting the x-axis in the angle α. The third side is the hyperbolic line whose Euclidean center $a + bi$ is the only solution to (23) that lies in the Euclidean sector determined by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Remark 4.6 *The location condition on $a + bi$ provides a simple criterion to select from multiple solutions returned from a numerical or other solution procedure of (23). Figure 6 illustrates the location condition and how the extraneous solutions to (23) correspond to other pencil intersections.*

Proof of Theorem 4.5. The only part of the theorem that does not immediately follow from the previous discussion is the location of $a + bi$ in the sector determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. To prove this, let H_B and H_C denote the branches of the hyperbolas $H(\overleftrightarrow{AB}, \beta)$ and $H(\overleftrightarrow{AC}, \gamma)$ that meet the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$; see Figure 6. From our earlier discussion on pencils the diameter elements of the pencils $\mathcal{P}^+(\overleftrightarrow{AB}, \beta)$, $\mathcal{P}^-(\overleftrightarrow{AB}, \beta)$, $\mathcal{P}^+(\overleftrightarrow{AC}, \gamma)$, and $\mathcal{P}^-(\overleftrightarrow{AC}, \gamma)$ all pass through A. This means that the lines in $\mathcal{P}^+(\overleftrightarrow{AB}, \beta)$ that meet the ray $\overrightarrow{AB}$ correspond to the points of H_B that lie above $\overrightarrow{AB}$. Correspondingly, the elements of $\mathcal{P}^-(\overleftrightarrow{AC}, \gamma)$ that meet $\overrightarrow{AC}$ must correspond to the points of H_C that lie below $\overrightarrow{AC}$. The point $a + bi$ corresponds to the unique element of $\mathcal{P}^+(\overleftrightarrow{AB}, \beta) \cap \mathcal{P}^-(\overleftrightarrow{AC}, \gamma)$ that meets both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus it must lie in between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. ■

Proof of Proposition 4.3. Let E_1 and E_2 (Figure 11) be the endpoints of ℓ' on the unit disc and let us determine the common lines in the intersection of the two pencils $\mathcal{P}^+(\ell', \beta)$ and $\mathcal{P}^-(\ell'', \gamma)$. Consider a variable line $\ell = \ell(x)$ in $\mathcal{P}^+(\ell', \beta)$ whose intersection point $\{x\} = \ell' \cap \ell(x)$ moves from E_1 to E_2 along ℓ'. This is also pictured in Figure 11 with $\beta = \pi/4$ and $\alpha = \pi/6$ (approximately). Motion along the line ℓ' induces an ordering $\succ$ of the points on ℓ' with $x \succ y$ meaning "x is later than y" or "x is to the *left* of y" in Figure 11. Let $e_1(x)$ and $e_2(x)$ be the two endpoints of $\ell(x)$. The two points $e_1(x)$ and $e_2(x)$ sweep out the boundary of the unit disc in opposite directions. Let us suppose that $e_1(x)$ travels in the counter-clockwise direction and $e_2(x)$ travels in the clockwise direction. In Figure 11, $e_1(x)$ sweeps across the top of the circle and $e_2(x)$ sweeps across the bottom. This sweeping occurs in a strictly increasing fashion. For otherwise, if one endpoint "backs up" then two distinct lines in the pencil either intersect or meet at the boundary, contradicting $i)$ of Proposition 3.3. Initially, for points on ℓ' near E_1, the line ℓ is contained in a small ball around E_1 and hence does not meet ℓ''. As we move along ℓ' one of the endpoints $e_1(x)$ or $e_2(x)$ reaches an endpoint of ℓ'' at $x = x_1$. The moving endpoints $e_1(x)$ and $e_2(x)$ cannot reach the endpoints of ℓ'' at the same time. For then $\ell'' \in \mathcal{P}^+(\ell', \beta)$ and so $\alpha = \pi - \beta$, contradicting our hypothesis $\alpha + \beta + \gamma < \pi$. Suppose, for the sake of argument, that $e_1(x_1)$ is the first point to reach ℓ''. Let us denote by x_2 the value for which $e_2(x)$ meets the endpoint of ℓ''.

For x satisfying $x_2 \succeq x \succeq x_1$, $\ell(x)$ meets ℓ'' and we denote by $\gamma(x)$ the clockwise angle from $\ell(x)$ to ℓ''. We are trying to find out how many x's satisfy $\gamma(x) = \gamma$. Obviously $\gamma(x_1) = 0$. Let x_i be the intersection point of ℓ' and ℓ. Then $\alpha + \beta + \gamma(x_i) = \pi$ since these angles form a straight angle at the intersection point x_i. Now as x moves along the line, the clockwise angle from $\ell(x)$ to ℓ'' must strictly increase until we reach x_i. To see this, suppose that we have intermediate points x_3 and x_4 satisfying $x_i \succ x_4 \succ x_3 \succ x_1$. The four lines $\ell', \ell'', \ell(x_3)$, and $\ell(x_4)$ determine a quadrilateral since the intersection point x_i is not between x_3 and x_4. The interior angles of this quadrilateral are β, $\pi - \beta$, $\pi - \gamma(x_4)$, and $\gamma(x_3)$ as we move clockwise around the quadrilateral. We must have $\beta + \pi - \beta + \pi - \gamma(x_4) + \gamma(x_3) < 2\pi$ or $\gamma(x_3) < \gamma(x_4)$. By monotonicity it follows that for any angle δ satisfying $0 < \delta < \gamma(x_i) = \pi - \alpha - \beta$ there is exactly one line $\ell(x)$ with $x_i \succ x \succ x_1$ and $\ell(x) \in \mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \delta)$. Once we pass x_i, the angle $\gamma(x)$ strictly decreases until $\gamma(x) = 0$, at $x = x_2$. Therefore, if $0 < \delta < \gamma(x_i) = \pi - \alpha - \beta$, then $\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \delta)$ has exactly two lines. Now it follows that $\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \gamma)$ always has exactly two lines because of the conditions $0 < \gamma$ and $\alpha + \beta + \gamma < \pi$. The remaining cases all follow from a simple modification of this proof. For example, if $\alpha + \beta < \gamma$ then $\pi - \gamma < \pi - \alpha - \beta$ and hence $\mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^-(\ell'', \pi - \gamma) = \mathcal{P}^+(\ell', \beta) \cap \mathcal{P}^+(\ell'', \gamma)$ contains two lines. This is illustrated in Figure 8. ■

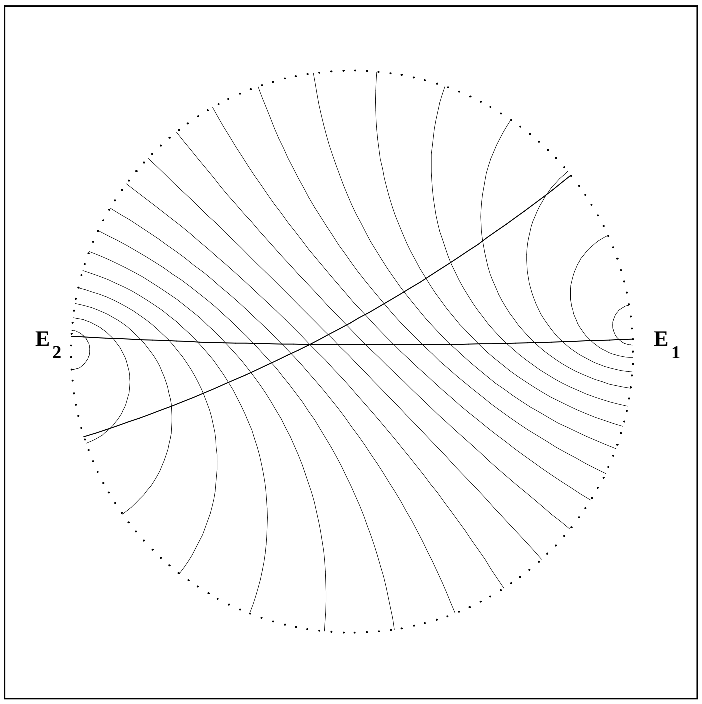

Fig. 11: A $\frac{\pi}{4}$-pencil with intersecting line.

5 Quadrilateral construction

In constructing quadrilaterals there is a new wrinkle: we must take side length into account. We wish to construct a quadrilateral $ABCD$ with prescribed angles at the corners. As in the case of the triangle we construct a standard quadrilateral from which the general quadrilateral can be constructed by a hyperbolic isometry. In our construction, our quadrilateral lies in the upper half of the disc with the side $\overline{AB}$ resting on the x-axis, centered at the origin. This quadrilateral is pictured in Figure 12. Suppose the angle information and the side length of $\overline{AB}$ is

$$\angle DAB = \alpha,\ \angle ABC = \beta,\ \angle BCD = \gamma,\ \angle CDA = \delta,$$

and

$$\rho(A, B) = h.$$

Here are the steps to construct the quadrilateral:

QC.1 Test to see if $\overline{AB}$ is long enough to guarantee that $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are disjoint, namely

$$\cosh h > \frac{\cos\alpha\cos\beta + 1}{\sin\alpha\sin\beta}.$$

QC.2 Define the points A and B on the x-axis by the formulas (see (3))

$$\frac{h}{2} = 2\tanh^{-1}(|A|) \text{ and } \frac{h}{2} = 2\tanh^{-1}(|B|).$$

Let $r = \tanh(h/4)$ so that $A = -r$ and $B = r$.

QC.3 Find the Euclidean centers of $\ell' = \overleftrightarrow{BC}$ and $\ell'' = \overleftrightarrow{AD}$. This is easily done if we know a point $x_0 + iy_0$ through which ℓ' passes and the slope dy/dx of ℓ' at $x_0 + iy_0$. In fact, for the point B we have $y_0 = 0$, $x_0 = r$ and $dy/dx = -\tan\beta$. Similarly for ℓ'' and for A, $y_0 = 0$, $x_0 = r$, and $dy/dx = \tan\alpha$. If $a + bi$ is the Euclidean center of ℓ' or ℓ'' then (7) gives

$$\begin{aligned} 2ax_0 + 2by_0 &= 1 + x_0^2 + y_0^2 \\ 2a + 2b\frac{dy}{dx} &= 2x_0 + 2y_0\frac{dy}{dx}, \end{aligned}$$

an easily solved set of linear equations.

QC.4 Find the Euclidean centre of the intersection of pencils $\mathcal{P}^+(\ell', \gamma) \cap \mathcal{P}^-(\ell'', \delta)$ by finding the intersection points of the corresponding hyperbolas. It turns out that there are four intersections of the hyperbolas. Each of these must be tested to find the line $\overleftrightarrow{CD}$ with a proper intersection with $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$. See Remark 3.6.

The hyperbolas, three of the four intersections, and the required line are shown in Figure 13.

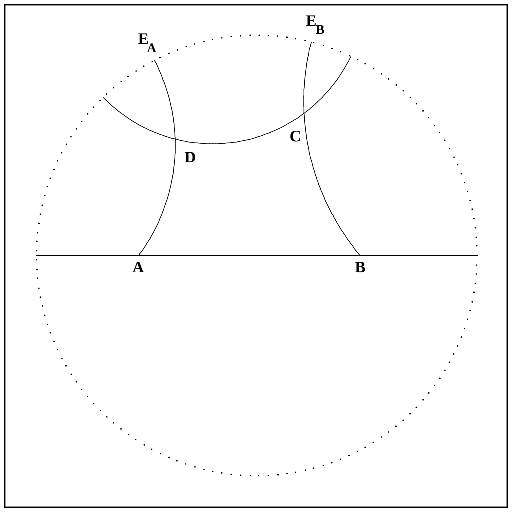

Fig. 12: A standard quadrilateral

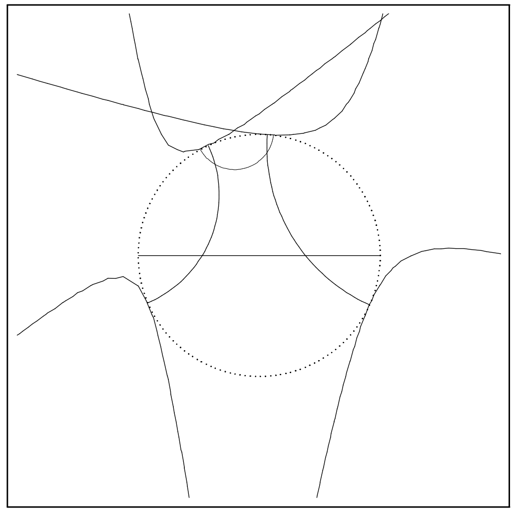

Fig. 13: Quadrilateral and hyperbolas

We formalize our discussion with the following theorem on quadrilateral construction.

Theorem 5.1 *Let α, β, γ, and δ be four angles satisfying $0 < \alpha, \beta, \gamma, \delta \leq \pi/2$ and $\alpha + \beta + \gamma + \delta < 2\pi$. Then, for every value of h satisfying*

$$\cosh h > \frac{\cos\alpha \cos\beta + 1}{\sin\alpha \sin\beta}$$

there is a quadrilateral $ABCD$, unique up to congruence, such that

$$\angle DAB = \alpha,\ \angle ABC = \beta,\ \angle BCD = \gamma,\ \angle CDA = \delta,$$

and such that the side length of $\overline{AB}$ satisfies

$$\rho(A, B) = h.$$

As in the case of triangles, we need a proposition about pencil intersections for pairs of disjoint lines, in order to prove our quadrilateral construction theorem. This proposition is much simpler than in the case of intersecting lines. The proof of this proposition is deferred to the end of the section.

Proposition 5.2 *Let ℓ' and ℓ'' be two parallel or disjoint lines in the hyperbolic plane, and let $0 < \gamma, \delta < \pi/2$. Then each of the four possible pencil intersections has exactly one point, i.e.,*

$$\begin{aligned} \left|\mathcal{P}^+(\ell',\gamma)\cap\mathcal{P}^+(\ell'',\delta)\right| &= \left|\mathcal{P}^+(\ell',\gamma)\cap\mathcal{P}^-(\ell'',\delta)\right| = 1 \\ \left|\mathcal{P}^-(\ell',\gamma)\cap\mathcal{P}^-(\ell'',\delta)\right| &= \left|\mathcal{P}^-(\ell',\gamma)\cap\mathcal{P}^+(\ell'',\delta)\right| = 1. \end{aligned} \tag{24}$$

The Euclidean centers of these lines are in intersections above one-to-one correspondence with the points of $H(\ell',\beta)\cap H(\ell'',\gamma))$ (including any intersections at infinity that correspond to diameter lines). If exactly one of or both of β and γ equals $\pi/2$ then the number of distinct sets is two or one, respectively. The intersections still contain only one line and they are in one-to-one correspondence to the points of intersection of $H(\ell',\beta)\cap H(\ell'',\gamma)$ that lie outside the unit circle.

Figure 14 shows an example of two disjoint lines ℓ' and ℓ'', the two hyperbola $H(\ell',\beta)$ and $H(\ell'',\gamma)$, the four points of intersection, and the four lines in the pencil intersections (24).

Proof of Theorem 5.1. To prove the theorem let us follow Steps QC.1-QC.4. The quadrilateral we are seeking is pictured in Figure 12. Assuming Step QC.1 has been verified, let ℓ_0 be the x-axis, pick $z_0 = r$ on ℓ_0 such that $2\tanh^{-1}(|z_0|) = h/2$, and set $A = -z_0$ and $B = z_0$, as in Step QC.2. Thus the hyperbolic length of $\overline{AB}$ is h. Now, as in Step QC.3, construct lines ℓ'' and ℓ' that meet ℓ_0 at A and B, respectively, and such that the following angle relations hold. Let E_A and E_B be the endpoints of ℓ'' and ℓ' on the upper half of the unit circle. We want $\angle BAE_A = \alpha$ and $\angle ABE_B = \beta$. We are assuming that h has been chosen large enough (Step QC.1) so that ℓ'' and ℓ' are disjoint lines. Now we seek a fourth side $\overline{CD}$ such that the line $\ell = \overleftrightarrow{CD}$ forms a counter-clockwise angle of measure γ with ℓ' and a clockwise angle of measure δ with ℓ''. That is, we are looking for $\mathcal{P}^+(\ell',\gamma)\cap\mathcal{P}^-(\ell'',\delta)$. Proposition 5.2 ensures that there is a unique line in this intersection, which may be found by Step QC.4. Next let us prove that $\overleftrightarrow{CD}$ lies in the upper half plane. If it lies in the lower half plane then the angle sum for $ABCD$ is $\pi-\alpha+\pi-\beta+\pi-\gamma+\pi-\delta \geq 2\pi$, since $\alpha,\beta,\gamma,\delta \leq \pi/2$. But the angle sum of a quadrilateral is less than 2π, yielding a contradiction. If C is in the upper half plane and D is in the lower half plane then $\overline{CD}$ must cross ℓ_0 at some point E on $\overline{AB}$. But then ΔEAD has angle sum $\pi-\alpha+\pi-\delta+\epsilon$ $= \pi + (\pi/2-\alpha)+ (\pi/2-\delta)+\epsilon$, where ϵ is the measure of $\angle AED$. But now this angle sum exceeds π, a contradiction. Similarly, we cannot have C

in the upper half plane and D in the lower half plane. Similar arguments eliminate the possibility that C or D lie on $\overline{AB}$. Thus both C and D and consequently $\overline{CD}$ lie in the upper half plane.

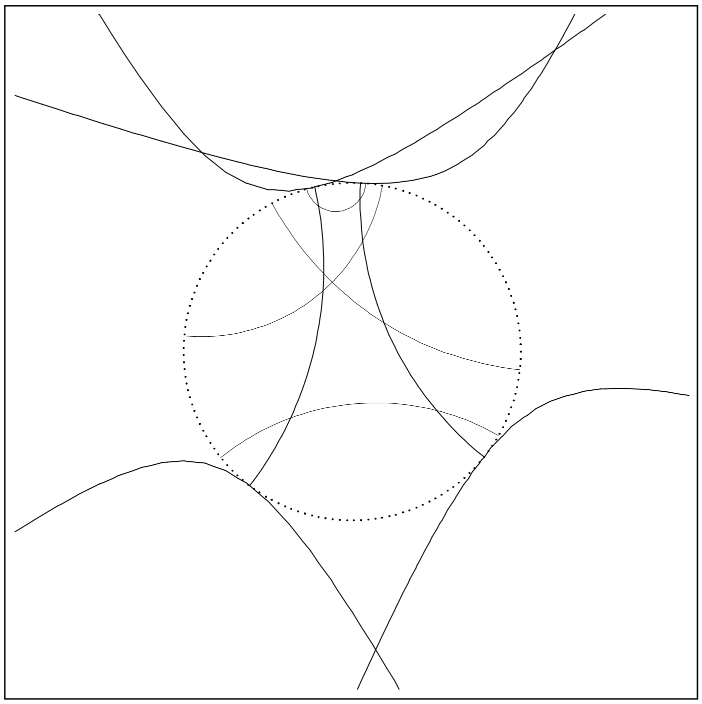

Fig. 14: Intersecting hyperbolas and pencils

Now let us suppose that ℓ' and ℓ'' are not disjoint and meet in point E, possibly on the boundary. By angle sum arguments this point must lie in the upper half plane. Now suppose it were possible for us to construct the required quadrilateral. Then C must lie in the interior of $\overline{BE}$ and D must lie in the interior of $\overline{AE}$. But then ΔCDE has angle sum $\pi - \gamma + \pi - \delta + \epsilon$, where ϵ is the measure of $\angle CED$. But then this angle exceeds π, a contradiction. Thus it is required to have ℓ' and ℓ'' disjoint. If ℓ' and ℓ'' do meet in E then the second law of cosines (5) gives us

$$\cosh h = \frac{\cos\alpha\cos\beta + \cos\varepsilon}{\sin\alpha\sin\beta}.$$

Since α and β are fixed the largest possible value of the right-hand side occurs when $\epsilon = 0$, i.e., E is on the boundary of the disc. For values of h

larger than

$$\cosh^{-1}\left(\frac{\cos\alpha\cos\beta+1}{\sin\alpha\sin\beta}\right)$$

the lines must be disjoint. ■

Proof of Theorem 5.2. The proof of this proposition is similar to that of Proposition 4.3. Thus we adopt the notation of that proof, except that x_2 is defined to be the value of x such that $e_1(x) = E_2$. The relevant picture in this case is Figure 15. We consider only the case $\mathcal{P}^+(\ell', \gamma) \cap \mathcal{P}^-(\ell'', \delta)$ as all other cases are similar. Let $\delta(x)$ be the counter-clockwise angle from $\ell = \ell(x)$ to ℓ''. Depending on which side of ℓ' the line ℓ'' is on, we either have $\delta(x)$ monotonically increasing from 0 at $x = x_1$ to π at $x = x_2$, or $\delta(x)$ monotonically decreasing from π at $x = x_1$ to 0 at $x = x_2$. In either case there is exactly one x_m with $x_2 \succ x_m \succ x_1$ and $\delta(x) = \delta$. Thus $\mathcal{P}^+(\ell', \gamma) \cap \mathcal{P}^-(\ell'', \delta)$ contains exactly one line. The centers of these lines must then be in one-to-one correspondence with the four points of intersection of $H(\ell', \gamma) \cap H(\ell'', \delta)$. ■

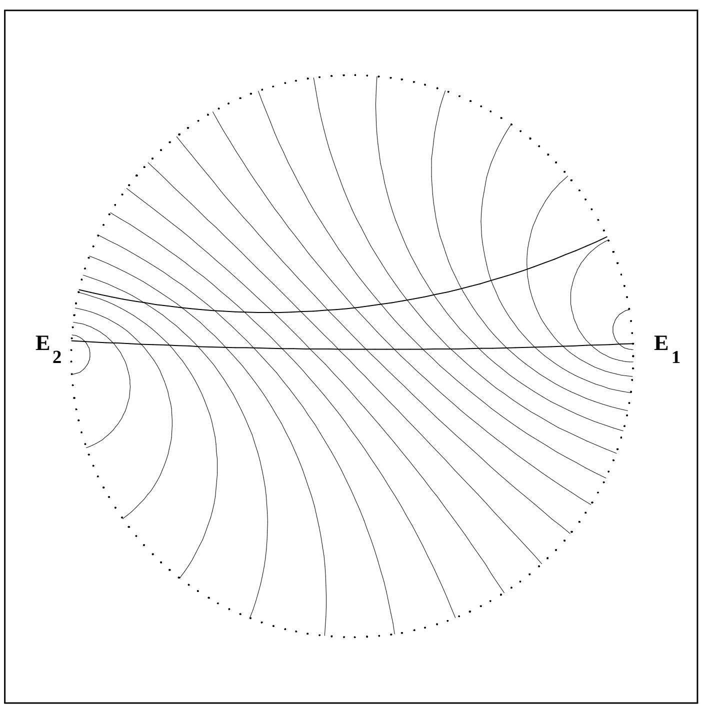

Fig. 15: A pencil and disjoint lines.

Remark 5.3 *If we are looking only for convex polygons we need to consider the case where* $\overleftrightarrow{AE_A}$ *and* $\overleftrightarrow{BE_B}$ *intersect in the upper half plane and* $\overleftrightarrow{CD}$ *meets these two lines below the intersection. In this case we need to use Proposition 4.3.*

6 Further questions

Here are some interesting questions suitable for homework or further research.

Homework

1. In (16) what happens when $\cos^2 \gamma$ is replaced by a number greater than 1? Following the proof of Proposition 3.3, it is easy to prove that the curve is an ellipse, tangent to the boundary circle at the endpoints of the line ℓ'. However, does the pencil have an interesting geometric interpretation?

2. Let $\mathcal{C}$ be a circle of radius R and center x_0 anywhere in the complex plane. What is the equation of the locus of hyperbolic lines tangent to $\mathcal{C}$ if tangencies outside of the unit disc are allowed? Note that if ℓ is a line with Euclidean radius r and centre z_0, then ℓ and $\mathcal{C}$ are tangent if and only if $|z_0 - x_0| = R + r$. The equation can be manipulated in a fashion similar to the equations in the proof of Proposition 3.2.

3. The same as question 2 except that the hyperbolic line meets $\mathcal{C}$ at an angle γ. A slight modification of the methods in Proposition 3.2 should work.

4. Develop the equations for elliptic, hyperbolic, and parabolic pencils [1, p. 170].

Further Research

5. Find a simple criterion to determine which of the four hyperbola intersections to choose in Step QC.4 of the quadrilateral construction.

6. Because of our parametrization, the clockwise and counter-clockwise angle pencils could not be cut out as a single algebraic curve. The hyperbola had to be split at the unit circle to distinguish the pencils, and the diameter lines were at infinity. In the space of geodesics $\mathcal{U}$

the pencils are separated. Here is the question: describe the various types of pencils (elliptic, parabolic, hyperbolic, angle pencils, and the additional pencils described in the homework section) as geometric objects on $\mathcal{U}$. It may first be useful to figure out how the group of isometries of $\mathbb{H}$ acts on $\mathcal{U}$.

7. Find a constructive procedure for kaleidoscopic pentagons, hexagons, etc. For the pentagon case we need all the work for the quadrilateral case in addition to figuring out when lines from two different angle pencils intersect and their angle of intersection. The ideas in the proof of Proposition 3.4 should work here. Two side lengths need to be specified.

8. Suppose that $\alpha = \pi/s, \ldots, \delta = \pi/v$, in the construction of a quadrilateral. Suppose also that the sidelength h is chosen so that the corresponding euclidean value $r = \tanh(h/4)$ is a rational number. Let Λ be the group of orientation preserving transformations in the reflection group generated in the sides of the constructed quadrilateral. As in Remark 4.1 it is easily shown that $\Lambda \subseteq PSL_2(K_r)$ for some number field K_r. How does K_r vary with r? The problem first needs to be stated in an invariant way, e.g., consider the quadrilateral up to rational congruence in $\mathbb{H}$.

Acknowledgments This work was motivated by work with undergraduates Dawn Haney, Lori McKeough, and Brandy Smith at the Rose-Hulman NSF-REU site (grant #DMS-9619714), see [2] and [3].

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