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Slip flow over a lubricated rotating disk

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Abstract

A set of slip-flow boundary conditions for the flow due to a lubricated disk rotating in a Newtonian fluid is derived. Similarity solutions are generally prohibited by the replacement of the conventional no-slip conditions by the new slip-flow conditions, except in the particular case when the power-law index of the non-Newtonian lubricant equals 1/3. The amount of velocity slip is controlled by a single dimensionless slip coefficient. Numerical solutions are presented for this case, showing that the three-dimensional flow field is dramatically affected by accentuated velocity slip. In particular, the axial flow towards the disk, i.e. the pumping efficiency, and the torque required to maintain steady rotation of the disk, decrease monotonically with increasing slip.

Keywords: Rotating disk flow; Slip conditions; Lubrication; Non-Newtonian fluids; Liquid film

1. Introduction

The flow of a viscous fluid which arises due to the rotation of a disk in an otherwise stagnant fluid constitutes a prototype of three-dimensional boundary layer problems. Von Kármán (1921) considered the steady laminar motion of an incompressible Newtonian fluid caused by a constantly rotating disk in a quiescent ambient. He devised an elegant similarity transformation which transforms the axi-symmetric Navier–Stokes equations into a set of ordinary differential equations (ODEs). Whereas von Kármán solved the resulting coupled set of non-linear ODEs by means of the momentum integral method approach, more accurate solutions have been provided by many others (e.g. Rogers and Lance, 1960). Moreover, due to its fundamental nature in combination with its practical importance, a vast number of modifications and extensions of the von Kármán's swirling flow problem exist. The review by Zandbergen and Dijkstra (1987) gives an excellent overview of the field.

Technical applications of rotating disk problems can be found for instance in viscometry, spin-coating, manufacturing and use of computer disks, and in various rotating machinery components. The flow arising due to the rotation of a single disk closely resembles the flow near the rotor in rotor–stator systems, which is of concern in air-cooled gas turbines. A comprehensive review and detailed discussions of flow phenomena occurring in such two-disk systems were provided by Owen and Rogers (1989).

Even the one-disk problem has received considerable attention over the years and different extensions of the now classic von Kármán problem have been made to address various applications. One such example is the extension of von Kármán's problem to a case in which the spinning disk is rotating in a non-Newtonian rather than in a Newtonian fluid. The flow is then no longer governed by the Navier–Stokes equations but by the Cauchy equations. Mitschka (1964) extended the von Kármán problem to disk flow in an inelastic power-law fluid. Andersson et al. (2001) and Denier and Hewitt (2004) have more recently reconsidered this particular problem and explored the difficulties that arise due to the severe non-linearity associated with the non-Newtonian rheological equation of state. In all these analyses, the simplified

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Cauchy equations were subjected to conventional no-slip boundary conditions at the disk surface.

A completely different extension of von Kármán's one-disk problem is the analysis of Sparrow et al. (1971). They considered the flow of a Newtonian fluid due to the rotation of a porous-surfaced disk and for that purpose replaced the conventional no-slip boundary conditions at the disk surface with a set of linear slip-flow conditions. A substantial reduction in torque then occurred as a result of surface slip. This problem was recently reconsidered by Miklavcic and Wang (2004) who pointed out that the same slip-flow boundary conditions as those used by Sparrow et al. (1971) also could be used for slightly rarefied gases or for flow over grooved surfaces.

The objective of the present study is to examine the laminar flow of a Newtonian bulk fluid arising from a solid rotating disk lubricated by a non-Newtonian liquid film. It is hypothesized that the apparent slip caused by the presence of a thin lubrication layer will have similar effects on the swirling bulk flow as the velocity slip caused by porosity of the disk material, e.g. to reduce the torque required to rotate the disk. First, however, we aim to deduce a new set of slip-flow boundary conditions for the lubricated disk problem. Thereafter, the influence of slip on the three-componential velocity field and, in particular, on the shaft torque, will be explored.

2. Slip-flow boundary conditions

2.1. Fluid motion inside the non-Newtonian lubrication layer

Let us assume that the steadily rotating disk is covered by a thin layer of an inelastic non-Newtonian liquid which obeys the power-law model due to Ostwald and deWaele. The stress tensor τ is related to the deformation rate tensor D by

$$\tau = 2\mu_L D = 2K(2D_{ij}D_{ij})^{\frac{(n-1)}{2}} D \tag{1}$$

where μ_L denotes the viscosity function. Here, K (kg m^{-1} s^{n-2}) is the consistency coefficient and n is the power-law index. This two-parameter rheological equation of state represents shear-thinning (pseudoplastic) fluids for $n < 1$ and shear-thickening (dilatant) fluids for $n > 1$. The special case $n = 1$ corresponds to Newtonian (i.e. linear) rheology with dynamic coefficient of viscosity K. Thus, the deviation of n from unity is equivalent with the degree of departure from Newtonian behaviour.

The solid disk rotates with a constant angular velocity Ω about the vertical z-axis and the center of the disk is at the origin of a cylindrical coordinate system (r, θ, z). The lubricant is introduced at a constant flow rate Q (m^3/s) through a point opening at the center of the infinitely large disk. The centrally introduced lubricant gradually spreads radially outwards and forms a thin lubrication layer of variable thickness $h(r)$. Due to the principle of mass conservation we readily obtain

$$Q = \int_0^{h(r)} U(r,z) \cdot 2\pi r \, dz \tag{2}$$

where $h(r)$ is the local thickness of the lubrication layer and $U(r,z)$ denotes the radial (i.e. outward) component of the velocity vector $V = [U, V, W]$ inside the lubricant. The circumferential velocity $V(r,z)$ does not contribute to the radially directed flow rate and the velocity component W perpendicular to the disk is negligible provided that the film is sufficiently thin.

Let us now follow Joseph (1980) who derived a slip-flow boundary condition for plane boundary layer flow over a lubricated surface. More recently, Joseph's slip flow condition was generalized by Andersson and Valnes (1999) to accommodate for the effect of a non-Newtonian lubricant. A basic assumption in these analyses is that lubrication theory applies, i.e. that the non-linear convective terms in the momentum equations are negligible. Moreover, the thickness $h(r)$ of the lubricating film was assumed sufficiently small so that also the streamwise pressure gradient could be neglected. The outcome of these assumptions is that a drag flow approximation applies. Now, by carrying Joseph's analysis over to the present axi-symmetric case, both the radial and the circumferential velocity components vary linearly from the surface of the disk $z = 0$ to the interface between the non-Newtonian lubricant and the Newtonian bulk fluid at $z = h(r)$, i.e.

$$U(r,z) = \frac{\widetilde{U}(r) \cdot z}{h(r)} \tag{3}$$

$$V(r,z) = \Omega r - \frac{(\Omega r - \widetilde{V}) \cdot z}{h(r)} \tag{4}$$

Here, $\widetilde{U}(r)$ and $\widetilde{V}(r)$ denote the interfacial velocity components, both in the lubricating liquid and in the Newtonian bulk fluid.

In order to eliminate the unknown thickness $h(r)$ of the lubrication layer, mass conservation (2) in combination with the linear velocity profile (3) gives

$$h(r) = \frac{Q}{\pi r \widetilde{U}(r)} \tag{5}$$

Now, the viscosity function μ_L in Eq. (1) simplifies considerably by assuming all velocity gradients but $\partial U/\partial z$ and $\partial V/\partial z$ to be negligible. This gives

$$\mu_L = K\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]^{\frac{(n-1)}{2}} = K\left(\frac{\pi}{Q}\right)^{n-1} \cdot (r\widetilde{U})^{n-1} \cdot \left[\widetilde{U}^2 + \left(\Omega r - \widetilde{V}\right)^2\right]^{\frac{(n-1)}{2}} \tag{6}$$

where the last equality is obtained by means of the linear velocity distributions (3) and (4).

The simplified expressions for the fluid motion (3) and (4) and the viscosity function (6) are applicable only within thin lubrication layers. The general equations of motions for an inelastic power-law fluid in thin shear layers can be found elsewhere (e.g. Mitschka, 1964; Andersson et al., 2001; Denier and Hewitt, 2004).

2.2. Interfacial matching

Both the velocity components and the shear stress components in the radial and circumferential directions should be continuous across the interface between the non-Newtonian lubricant and the Newtonian bulk fluid. Thus, $U = u$ and $V = v$ at $z = h(r)$, where u and v denote the bulk fluid velocity in the r and θ directions, respectively. Similarly, by requiring continuity of the shear stress components at $z = h(r)$, we obtain that

$$\mu \frac{\partial u}{\partial z} = \mu_L \frac{\partial U}{\partial z} \approx \mu_L \frac{\widetilde{U}}{h} \tag{7}$$

$$\mu \frac{\partial v}{\partial z} = \mu_L \frac{\partial V}{\partial z} \approx -\mu_L \frac{(\Omega r - \widetilde{V})}{h} \tag{8}$$

where the left-hand sides represent the shear stresses in the bulk fluid and the right sides express the stress components in the lubricating fluid. With the simplified expression (6) for the viscosity function μ_L, the interfacial conditions (7) and (8) become

$$\frac{\partial u}{\partial z} = \frac{K}{\mu}\left(\frac{\pi}{Q}\right)^n u(ru)^n \left[u^2 + (\Omega r - v)^2\right]^{\frac{(n-1)}{2}} \tag{9}$$

$$\frac{\partial v}{\partial z} = -\frac{K}{\mu}\left(\frac{\pi}{Q}\right)^n (\Omega r - v)(ru)^n \left[u^2 + (\Omega r - v)^2\right]^{\frac{(n-1)}{2}} \tag{10}$$

These expressions, together with $w = 0$, can be considered as boundary conditions for the bulk flow $\boldsymbol{v} = [u, v, w]$ at $z = h(r)$. Following Joseph (1980), however, the boundary conditions (9) and (10) for the bulk flow can be imposed at the disk $z = 0$ rather than at $z = h(r)$ since the thickness $h(r)$ is assumed to be very small.

The above Eq. (9) is the axi-symmetric equivalent of the slip-flow condition for planar bulk flow deduced by Andersson and Valnes (1999). In that case of a two-dimensional boundary layer flow past a lubricated surface, the amount of slip u is directly related to the associated shear rate $\partial u/\partial z$. In the present case of a three-dimensional boundary layer flow, the boundary conditions (9) and (10) couple the motion u in the radial direction to the velocity v in the circumferential direction and vice versa.

For the particular case of a Newtonian lubricant, i.e. $n = 1$, the above expressions simplify to

$$\frac{\partial u}{\partial z} = \frac{K}{\mu}\frac{\pi}{Q} ru^2 \tag{11}$$

$$\frac{\partial v}{\partial z} = -\frac{K}{\mu}\frac{\pi}{Q} ru(\Omega r - v) \tag{12}$$

Here, K is now the dynamic coefficient of viscosity which is measured in (kg m^{-1} s^{-1}). Analogous slip-flow boundary conditions were recently used by Solbakken and Andersson (2004) to mimic the effect of a lubricating film on the turbulent bulk flow in a plane channel.

3. Bulk flow driven by a lubricated disk

3.1. Similarity considerations

The steady motion of the Newtonian bulk fluid is governed by the incompressible axi-symmetric Navier–Stokes equations. By means of the now classic similarity transformation

$$z^* = z\sqrt{\frac{\Omega}{\nu}} \tag{13}$$

$$u = \Omega r f(z^*); \quad v = \Omega r g(z^*); \quad w = \sqrt{\Omega \nu} h(z^*); \quad p = \Omega \mu \cdot p^*(z^*) \tag{14}$$

it is assumed that u/r, v/r, w and p are functions of z^* only (see e.g. Von Kármán, 1921; Rogers and Lance, 1960). If so, mass continuity and the governing equations of motion transform into the following set of non-linear ordinary differential equations (ODEs):

$$h' = -2f \tag{15}$$

$$f'' = f^2 - g^2 + hf' \tag{16}$$

$$g'' = 2fg + hg' \tag{17}$$

$$p^{*\prime} = 2fh - 2f' \tag{18}$$

where the prime denotes differentiation with respect to the non-dimensional axial coordinate z^*.

In the usual case of a dry disk, for which the conventional no-slip conditions $f(0) = 0$ and $g(0) = 1$ apply for the radial and circumferential velocity components, respectively, similarity is achieved. In the case of a lubricated disk, on the other hand, similarity is generally prohibited since the transformations (13) and (14) do not remove the explicit appearance of r from the slip-flow boundary conditions (9) and (10). Therefore, h, f, g and p^* must be treated as functions of both z^* and r and the ODEs (15)–(18) are no longer valid.

For the particular parameter value $n = 1/3$, however, the slip-flow conditions (9) and (10) transform exactly into

$$f'(0) = \lambda [f(0)]^{4/3} \left[(f(0))^2 + (1 - g(0))^2\right]^{-1/3} \tag{19a}$$

$$g'(0) = -\lambda [f(0)]^{1/3} [1 - g(0)] \left[(f(0))^2 + (1 - g(0))^2\right]^{-1/3} \tag{19b}$$

and true similarity is achieved. These conditions, together with the impermeability and pressure condition

$$h(0) = 0 \tag{19c}$$

$$p^*(0) = 0 \tag{19d}$$

at the disk and the outer conditions

$$f = g = 0 \quad \text{as } z^* \to \infty \tag{20}$$

make the two-point boundary-value problem completely defined. Here, the boundary conditions (20) imply that both the radial (f) and the azimuthal (g) motion vanish sufficiently far away from the rotating disk, whereas the axial velocity component (h) is anticipated to approach a yet unknown asymptotic limit for sufficiently large z^*-values.

The only parameter present in the transformed slip-flow problem is the dimensionless slip coefficient

$$\lambda \equiv \frac{L_{\text{visc}}}{L_{\text{lub}}} = \frac{\sqrt{\nu/\Omega}}{\left(\frac{\mu}{K}\right)\left(\frac{Q\Omega}{\pi}\right)^{1/3}} \tag{21}$$

which can be interpreted as the ratio between a viscous length scale L_{visc} and a lubrication length L_{lub}. It should be emphasized that the latter length scale is representative only for power-law fluids with $n = 1/3$. By using a small amount Q of a highly viscous (i.e. large K) lubricant, the lubrication length L_{lub} is small and the slip coefficient λ correspondingly large. In the limit as λ tends to infinity, the conventional no-slip conditions $f(0) = 0$ and $g(0) = 1$ are recovered from (19a,b). On the contrary, if the lubrication length L_{lub} becomes infinitely large the slip coefficient λ vanishes and full slip $f'(0) = g'(0) = 0$ is achieved.

3.2. Similarity solutions for n = 1/3

Although the new set of slip-flow boundary conditions (9) and (10) are applicable for bulk flow over a lubricated disk, irrespective of the value of the power-law index n of the lubricant, a similarity problem can only be achieved for a shear-thinning lubricant with $n = 1/3$. We purposely considered this particular parameter value to illustrate the slip effect of a lubrication layer. This choice was motivated by the fact that a set of ODEs can easily be solved to practically any desired degree of accuracy. For any other parameter value than $n = 1/3$, true similarity solutions are prohibited by the appearance of an explicit r-dependence in the transformed slip-flow boundary conditions. In such cases, we are left with a set of PDEs rather than ODEs. In spite of the formal motivation for the choice $n = 1/3$, it is worth to remember that many non-Newtonian liquids have power-law indices n close to 0.3 (see e.g. Andersson and Irgens, 1990).

The governing ODEs (15)–(18) were written as a system of first-order equations and integrated numerically by a standard Runge–Kutta technique for a variety of values of the slip coefficient λ in the range from 0.01 to 10.0. First, however, the present integration technique was used to solve the slip-flow problem considered by Miklavcic and Wang (2004) and the results obtained reproduced the results provided in their Table 1 to within 0.01%.

Similarity solutions for the three velocity components and the pressure are presented in Fig. 1 for some different values of λ. The solution for the no-slip case $\lambda \to \infty$, which was obtained with the conventional no-slip boundary conditions $f(0) = 0$ and $g(0) = 1$, serves as a reference. Although the results are shown only from the disk $z^* = 0$ to $z^* = 8.0$, the numerical integrations were performed over a substantially larger domain in order to assure that the outer boundary conditions (20) were satisfied. The shear-driven motion (g) in the circumferential direction in Fig. 1(b) is gradually reduced with decreasing values of λ, i.e. with an increasing amount of slip. The centrifugal force associated with this circular motion causes the outward radial flow (f), which is correspondingly reduced with decreasing slip coefficient. The radial outflow f is compensated by an axial inflow $-h$ towards the rotating disk, in accordance with the mass conservation equation (15). The accompanying pressure in the vicinity of the rotating disk is higher than the ambient pressure and the axial flow is therefore decelerated by an adverse pressure gradient $p^{*\prime} < 0$ as the disk is approached (see Fig. 1d). The magnitude of the axial pressure gradient reduces with increasing slip (i.e. decreasing λ-values). These overall tendencies are fully consistent with results presented by Sparrow et al. (1971) and Miklavcic and Wang (2004) for flow due to either porous or rough-surfaced rotating disks. In their contexts, however, the slip velocities $f(0)$ and $1 - g(0)$ were linearly related to the corresponding shear rates $f'(0)$ and $g'(0)$, whereas highly non-linear and strongly coupled slip-flow conditions have been used in the present study. Quantitative comparisons between the present results and those of Sparrow et al. (1971) and Miklavcic and Wang (2004) do therefore not make sense.

It is nevertheless interesting to notice that both Sparrow et al. (1971) and Miklavcic and Wang (2004) observed that the radial velocity reached a maximum near $z^* = 0.92$ in the no-slip case and that the effect of slip was to reduce the peak level of f and shift its location closer to the disk. In the present study, however, the highest peak level $f_{\max} = 0.1817$ is observed for $\lambda = 10$ and the location of the peak is close to $z^* = 0.90$, i.e. slightly closer to the disk than in the no-slip case although the peak level slightly exceeds the velocity peak 0.1806 found in the no-slip case. The variation of the peak level $f_{\max}$ of f is shown in Fig. 2(a) over the λ-range from 0.01 to 10.0, together with the variation of the radial slip velocity $f(0)$. Maximum slip $f(0) = 0.1283$ is achieved for λ close to 1. For slip coefficients $\lambda < 1$, both the slip velocity and the peak level of the radial velocity profile decrease as λ is further reduced. This behaviour is completely different from that of the circumferential velocity component. Here, the slip velocity $1 - g(0)$ shown in Fig. 2(b) increases monotonically from close to zero (i.e. practically no-slip) to close to unity (i.e. practically full slip) as the slip coefficient λ decreases from 10 to 0.01. Results for the limiting cases $\lambda = 0$ and $\lambda \to \infty$ are not included in Fig. 2. For the lowest parameter value $\lambda = 0.01$, the disk values $f(0) = 0.0060$ and $g(0) = 0.0146$ are obtained.

The gradual reduction of the peak in the f-profiles in Fig. 1(a) with decreasing values of λ is reflected in the distributions of the axial velocity component in Fig. 1(c). The distinct inflection point in the h-profiles for the highest values of λ seems to gradually disappear with increasing slip. This is a consequence of the direct coupling between the radial and axial velocity components through the continuity constraint (15). The reduction of the radial velocity $f(z^*)$ with decreasing λ automatically gives rise to a reduced axial inflow since

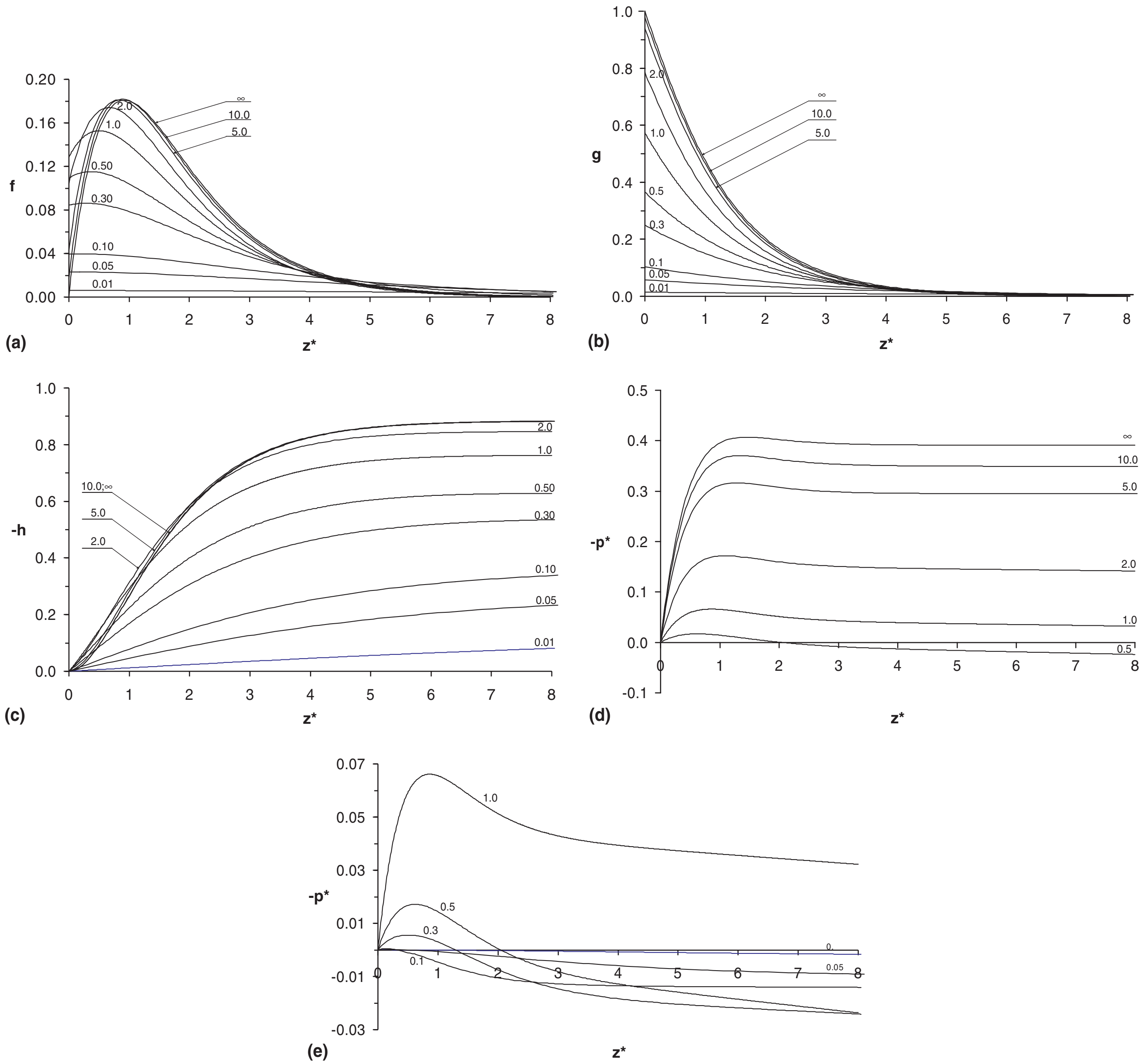

Fig. 1. Similarity solutions for a lubricant with $n = 1/3$ for some different values of the slip coefficient λ. (a) Radial velocity component $f(z^*)$, (b) circumferential velocity component $g(z^*)$, (c) axial velocity component $-h(z^*)$, (d) pressure $-p^*(z^*)$ for $\lambda > 0.5$ and (e) pressure $-p^*(z^*)$ for $\lambda < 1.0$.

$$-h(\infty) = 2\int_0^\infty f \cdot \mathrm{d}z^* \tag{22}$$

The results in Fig. 2(b) show that $-h(\infty)$ is surprisingly close to its no-slip limit 0.883 over a fairly large range of λ-values.

A closer look at the pressure distributions in Fig. 1(e) reveals a change of sign of pressure for the lowest values of the slip coefficient. This anomalous behaviour is clearly reflected in Fig. 2(c), which shows that the ambient pressure $-p^*(\infty)$ gradually approaches the pressure $p^*(0) = 0$ at the disk itself. For the lowest λ-values, however, the ambient pressure exceeds the disk pressure, which means that the axial pressure gradient drives the flow towards the disk in this particular parameter range, whereas the axial motion is normally retarded by the high pressure zone adjacent to the disk. The explanation of this unexpected phenomenon can be sought in the axial momentum equation (18). Here, f and f' can be replaced by h' and h'' and the resulting equation for $p^{*\prime}$ integrated once to give

$$-p^*(z^*) = 2[f(z^*) - f(0)] + \frac{1}{2}h^2(z^*) \tag{23}$$

where $f(0) = 0$ in the no-slip case. Then, since f always vanishes at infinitely large distances from the disk, cf Eq. (20),

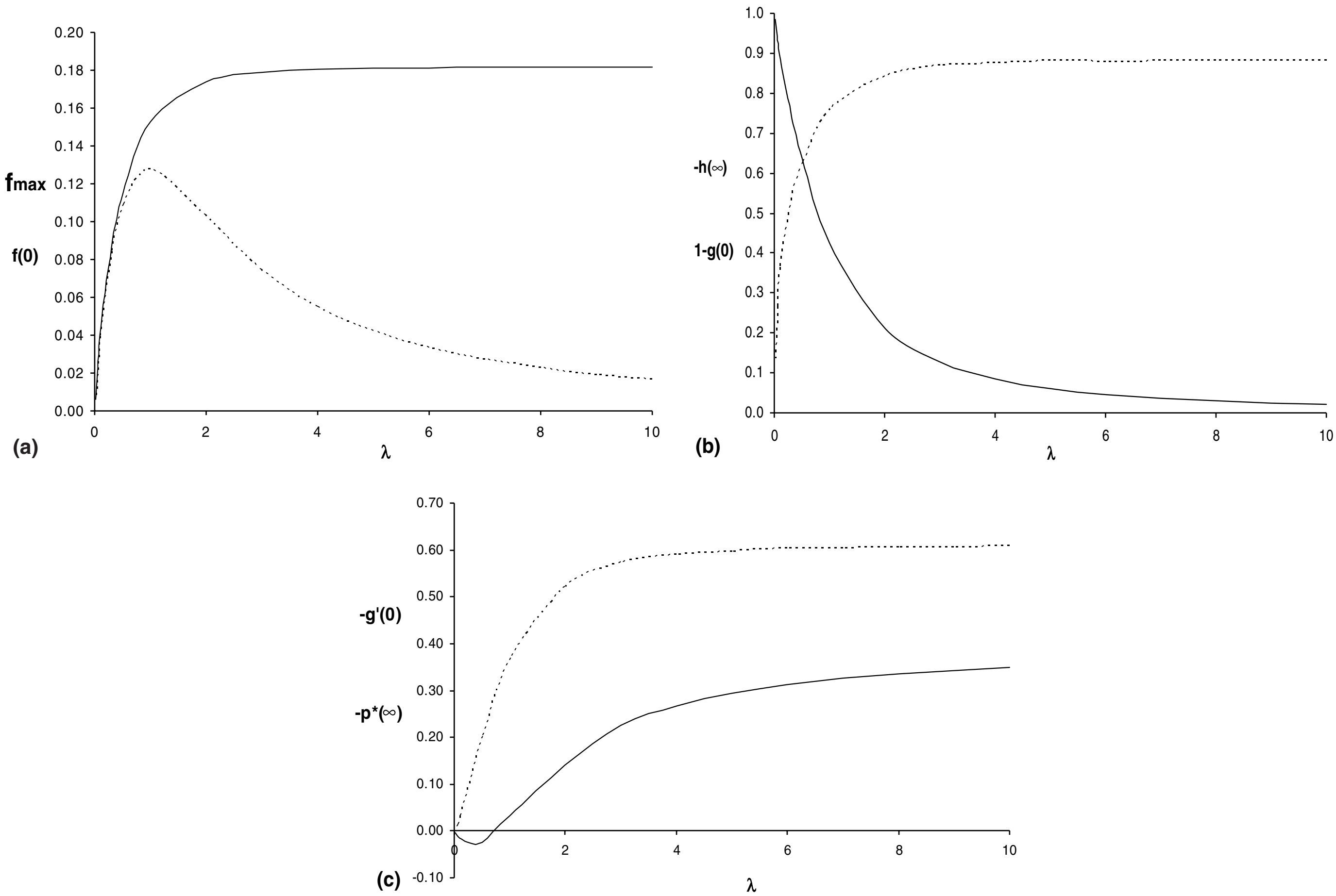

Fig. 2. Effect of slip coefficient λ on some flow characteristics ($n = 1/3$). (a) Radial slip velocity $f(0)$ (broken line) and f_{max} (solid line), (b) circumferential slip velocity $1 - g(0)$ (solid line) and asymptotic axial inflow $-h(\infty)$ (broken line), (c) ambient pressure $-p^*(\infty)$ (solid line) and wall shear rate $-g'(0)$ (broken line).

the ambient pressure $-p^*(\infty)$ is determined solely by the axial inflow velocity $h(\infty)$. In the present case with partial slip, however, the positive contribution by axial convection to the right hand side of (23) is partly outweighed by viscous normal stresses, i.e. $-2f'$ or h'' in Eq. (18). The rapid decrease of the magnitude of the axial inflow $-h(\infty)$ with decreasing slip coefficient for λ below 2 (Fig. 2(b)), in combination with a substantial slip velocity $f(0)$, explains the observed sign reversal of $-p^*(\infty)$ in Fig. 2(c). Neither Sparrow et al. (1971) nor Miklavcic and Wang (2004) reported a similar phenomenon.

The torque required to maintain steady rotation of the disk is determined by the circumferential shear stress component at the disk surface. Implicit in the drag flow approximations (3) and (4) is the constancy of the shear stress components across the thin lubrication layer. The circumferential shear stress exerted by the lubricant on the solid disk is therefore equal to the circumferential shear stress exerted by the bulk fluid on the lubrication layer, i.e. to the left-hand side of Eq. (8). Thus, the variation of the wall shear rate $-g'(0)$ with λ in Fig. 2(c) implies that the imposed torque decreases monotonically with increasing slip.

4. Concluding remarks

Newtonian slip flow due to a lubricated rotating disk has been examined for the first time. The new set of slip-flow boundary conditions aimed to accommodate for the partial slip effect caused by a thin lubricating layer is applicable for any value of the power-law index n, including the case of a Newtonian lubricant with $n = 1$. Results are obtained only for the particular parameter value $n = 1/3$, which is the only value for which similarity is achieved. For other values of the power-law index, the governing set of equations of motion do not transform to ODEs.

The computed similarity solutions for $n = 1/3$ show that the spin-up by viscous action of the bulk fluid next to the rotating disk is reduced with increasing slip. The radially directed centrifugal force is correspondingly reduced. The radial slip turned out to be insufficient to outweigh the reduced centrifugal force. The radial slip velocity $f(0)$ therefore reaches a maximum value slightly above 0.12 when the slip coefficient λ is close to unity, i.e. when the lubrication length L_{lub} and the viscous length scale L_{visc} nearly equals. An unexpected reversal of the axial pressure gradient is detected for the lower values of the slip coefficient.

The analysis presented herein, and thus the computed solutions, are based on the following assumptions. First of all, the lubricant is assumed to obey the non-linear Ostwald–deWaele power-law model as many non-Newtonian substances do, in particular under simple shear flow conditions. Next, it is essential for the new slip-flow conditions (9) and (10) to apply that the lubricating film is sufficiently thin so that lubrication theory applies, i.e. that the convection terms in the equations of motion are negligible. The bulk flow, on the other hand, is governed by the incompressible Navier–Stokes equations which are valid for low-speed motion of Newtonian (i.e. linear) fluids and these equations transform exactly into the ODEs (15)–(18) only for a lubricant with $n = 1/3$. Throughout the analysis, we have moreover assumed that the flow is steady, laminar and axisymmetric. It is well known that a non-axisymmetric instability arises at moderate rotation rates (see e.g. Gregory et al. (1955) for a pioneering study). The present results are therefore valid only for a lower range of the angular velocity Ω. Moreover, at high rotation rates a transition to turbulence will occur at a certain radial distance from the axis of rotation and the present analysis is no longer valid.

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