1.3 Example 3

Solve

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0 \tag{1}$$

Using the transformation

$$\eta = \int e^{-\int f(z)dz} dz$$

But

$$\begin{aligned}\frac{dw}{dz} &= \frac{dw}{d\eta}\frac{d\eta}{dz}\\ &= \frac{dw}{d\eta}e^{-\int f(z)dz} \end{aligned}\tag{2}$$

And

$$\begin{aligned}\frac{d^2w}{dz^2} &= \frac{d}{dz}\left(\frac{dw}{d\eta}e^{-\int f(z)dz}\right)\\ &= \left(\frac{d}{dz}e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-\int f(z)dz}\left(\frac{d}{dz}\frac{dw}{d\eta}\right)\\ &= \left(-f(z)e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-\int f(z)dz}\left(\frac{\frac{d}{d\eta}}{\frac{dz}{d\eta}}\frac{dw}{d\eta}\right)\\ &= \left(-f(z)e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-\int f(z)dz}\left(\frac{d\eta}{dz}\frac{d}{d\eta}\frac{dw}{d\eta}\right)\\ &= \left(-f(z)e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-\int f(z)dz}\left(\frac{d\eta}{dz}\frac{d^2w}{d\eta^2}\right)\end{aligned}$$

But $\frac{d\eta}{dz} = e^{-\int f(z)dz}$. The above becomes

$$\begin{aligned}\frac{d^2w}{dz^2} &= \left(-f(z)e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-\int f(z)dz}\left(e^{-\int f(z)dz}\frac{d^2w}{d\eta^2}\right)\\ &= \left(-f(z)e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-2\int f(z)dz}\left(\frac{d^2w}{d\eta^2}\right)\end{aligned}\tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}\left(-f(z)e^{-\int f(z)dz}\right)\frac{dw}{d\eta} + e^{-2\int f(z)dz}\left(\frac{d^2w}{d\eta^2}\right) + f(z)\frac{dw}{d\eta}e^{-\int f(z)dz} + g(z)w &= 0\\ e^{-2\int f(z)dz}\left(\frac{d^2w}{d\eta^2}\right) + g(z)w &= 0\\ \frac{d^2w}{d\eta^2} + g(z)e^{2\int f(z)dz}w &= 0\end{aligned}$$