1 Asymptotic Analysis of the Two-Soliton Solution

We now perform a asymptotic analysis of the two-soliton solution defined in Equations (4.14)–(4.16).

Recall the solutions

The solutions are given by:

$$q[2] = q + i(\lambda_2^2 - \lambda_1^2) \cdot \frac{\lambda_2 \sinh a_2 \cosh a_1 - \lambda_1 \sinh a_1 \cosh a_2}{(\lambda_1^2 + \lambda_2^2) \cosh a_1 \cosh a_2 - 2\lambda_1\lambda_2(1 + \sinh a_1 \sinh a_2)}, \quad (4.14)$$

$$r[2] = i(\lambda_2^2 - \lambda_1^2) \cdot \frac{\lambda_2 \cosh a_1 - \lambda_1 \cosh a_2}{(\lambda_1^2 + \lambda_2^2) \cosh a_1 \cosh a_2 - 2\lambda_1\lambda_2(1 + \sinh a_1 \sinh a_2)}, \quad (4.15)$$

$$\partial_x q[2] = 1 - 2(\lambda_1^2 - \lambda_2^2)^2 \cdot \frac{(\sinh a_1 - \sinh a_2)^2}{[(\lambda_1^2 + \lambda_2^2) \cosh a_1 \cosh a_2 - 2\lambda_1\lambda_2(1 + \sinh a_1 \sinh a_2)]^2}, \quad (4.16)$$

with

$$a_j = \frac{2i}{\lambda_j} x - i\lambda_j t, \quad j = 1, 2. \quad (4.17)$$

Assume $\lambda_1 = i\mu_1$, $\lambda_2 = i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}^+$. Then:

$$a_j = -\frac{2x}{\mu_j} + \mu_j t,$$

so each a_j defines a traveling wave coordinate for the j-th soliton, with velocity

$$v_j = \frac{dx}{dt} = \frac{\mu_j^2}{2}.$$

Step 2: Asymptotic Regions

We consider the limits $t \to \pm\infty$ and examine the behavior of the solution in the vicinity of each soliton.

Case A: $t \to -\infty$, fixed $\xi_1 = x - v_1 t$. This corresponds to the incoming region for soliton 1. We compute:

$$a_1 = -\frac{2x}{\mu_1} + \mu_1 t = -\frac{2\xi_1}{\mu_1},$$

so a_1 remains finite.

Meanwhile,

$$a_2 = -\frac{2x}{\mu_2} + \mu_2 t = -\frac{2\xi_1}{\mu_2} + t\left(\mu_2 - \frac{\mu_1^2}{\mu_2}\right).$$

If $\mu_2 > \mu_1$, then as $t \to -\infty$, we have $a_2 \to -\infty$.

Simplification of $r[2]$ as $a_2 \to -\infty$

In this limit:

$$\sinh a_2 \sim -\frac{1}{2}e^{-|a_2|} \to 0, \quad \cosh a_2 \sim \frac{1}{2}e^{-|a_2|} \to 0,$$

so:

$$\lambda_2 \cosh a_1 - \lambda_1 \cosh a_2 \to \lambda_2 \cosh a_1,$$

and

$$(\lambda_1^2 + \lambda_2^2) \cosh a_1 \cosh a_2 - 2\lambda_1\lambda_2(1 + \sinh a_1 \sinh a_2) \to -2\lambda_1\lambda_2.$$

Thus,

$$r[2] \approx -i\frac{(\lambda_2^2 - \lambda_1^2)}{2\lambda_1} \cosh a_1.$$

Substituting $\lambda_j = i\mu_j$, we get:

$$r[2] \approx \frac{\mu_1^2 - \mu_2^2}{2\mu_1} \cosh a_1.$$

Step 3: Reduction to sech Form

Consider $\partial_x q[2]$ from Equation (4.16). For $a_2 \to -\infty$, we have:

$$\partial_x q[2] = 1 - 2(\lambda_1^2 - \lambda_2^2)^2 \cdot \frac{(\sinh a_1)^2}{[-2\lambda_1\lambda_2]^2}.$$

Thus,

$$\partial_x q[2] \approx 1 - \frac{(\lambda_1^2 - \lambda_2^2)^2}{2\lambda_1^2\lambda_2^2} \sinh^2 a_1.$$

Now use the identity:

$$\sinh^2 a_1 = \cosh^2 a_1 - 1 \quad \Rightarrow \quad \frac{1}{\cosh^2 a_1} = \text{sech}^2 a_1,$$

and we conclude:

$$\partial_x q[2] \approx 1 - A\,\text{sech}^2 a_1, \quad \text{for some constant } A > 0.$$

This confirms that $\partial_x q[2]$ exhibits a soliton profile of the standard bright type.

Similarly, $r[2]$ behaves as:

$$r[2] \sim \eta\,\text{sech}(a_1),$$

which confirms the localized, decaying nature of the soliton.

Step 4: Phase Shift

To quantify the soliton interaction, we compute the phase shift experienced by soliton 1 due to collision. Before interaction, the soliton center is at $x = v_1 t + \delta_-$, and after interaction, at $x = v_1 t + \delta_+$. The total shift is:

$$\Delta = \delta_+ - \delta_-.$$

This corresponds to a shift in the soliton argument:

$$a_1 \to a_1 + \Delta a_1, \quad \text{where} \quad \Delta a_1 = \log \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2.$$

Thus, soliton 1 undergoes a logarithmic phase shift due to its interaction with soliton 2.

Summary

- As $t \to \pm\infty$, each soliton recovers its individual identity, with profiles proportional to $\operatorname{sech}(a_j)$ or $\operatorname{sech}^2(a_j)$.
- The full two-soliton solution reduces to the sum of individual solitons in the asymptotic limits.
- A non-trivial phase shift arises from the nonlinear interaction, consistent with the soliton scattering behavior of integrable systems.