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Theory of Fractional Order Generalized Thermoelasticity

In this work, a new model of thermoelasticity theory has been constructed in the context of a new consideration of heat conduction with fractional order, and its uniqueness theorem has been approved also. One-dimensional application for a half-space of elastic material, which is thermally shocked, has been solved by using Laplace transform and state-space techniques. According to the numerical results and its graphs, conclusion about the new theory of thermoelasticity has been constructed.
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1 Introduction

Recently, a considerable research effort has been expended to study anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation by Kimmich [1]

$$\rho c = \kappa I^{\alpha} c_{,ii} \tag{1}$$

where ρ is the mass density, c is the concentration, κ is the diffusion conductivity, i is the coordinate symbol, which takes the values 1, 2, and 3, the subscript "," means the derivative with respect to x_i, and notion I^{α} is the Riemann–Liouville fractional integral is introduced as a natural generalization of the well-known n-fold repeated integral $I^n f(t)$ written in a convolution-type form as in Refs. [2,3]

$$\left.\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \\ I^0 f(t) &= f(t) \end{aligned}\right\} \quad 0 < \alpha \leq 2 \tag{2}$$

where $\Gamma(\alpha)$ is the gamma function.

According to Kimmich [1] Eq. (1) describes different cases of diffusion where $0<\alpha<1$ corresponds to weak diffusion (subdiffusion), $\alpha=1$ corresponds to normal diffusion, $1<\alpha<2$ corresponds to strong diffusion (superdiffusion), and $\alpha=2$ corresponds to ballistic diffusion.

It should be noted that the term diffusion is often used in a more generalized sense including various transport phenomena. Equation (1) is a mathematical model of a wide range of important physical phenomena, for example, the subdiffusive transport occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous, and random media. Superdiffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, transport of organic molecules and atomic clusters on surface [4]. One might expect the anomalous heat conduction in media where the anomalous diffusion is observed.

Fujita [5,6] considered the heat wave equation for the case of $1 \leq \alpha \leq 2$

$$\rho C T = k I^{\alpha} T_{,ii} \tag{3}$$

where C is the specific heat, k is the thermal conductivity, and the subscript "," means the derivative with respect to the coordinates x_i.

Equation (3) can be obtained as a consequence of the non local constitutive equation for the heat flux components q_i is in the form

$$q_i = -k I^{\alpha-1} T_{,i}, \quad 1 < \alpha \leq 2 \tag{4}$$

Povstenko [4] used the Caputo heat wave equation defined in the form

$$q_i = -k I^{\alpha-1} T_{,i}, \quad 0 < \alpha \leq 2 \tag{5}$$

to get the stresses corresponding to the fundamental solution of a Cauchy problem for the fractional heat conduction equation in one-dimensional and two-dimensional cases.

Some applications of fractional calculus to various problems of mechanics of solids are reviewed in the literature [7,8].

2 Theory of Fractional Order Generalized Thermoelasticity

The classical thermoelasticity is based on the principles of the theory of heat conduction, which is called the Fourier law, which relates the heat flux components q_i to the temperature gradient as follows:

$$q_i = -k T_{,i} \tag{6}$$

In combination with the energy conservative law, this leads to the parabolic heat conduction equation, which is considered by Povstenko [4]

$$\rho C \dot{T} = k T_{,ii} \tag{7}$$

where the dotted above T means the derivative with respect to the time t.

Recently, in the nonclassical thermoelasticity theories, Fourier law (6) and heat conduction (7) are replaced by more general equations; these have been formulated. The first well-known generalized of such a type is that of Lord and Shulman [9], and it takes the form

$$q_i + \tau_o \dot{q}_i = -k T_{,i} \tag{8}$$

which leads to the hyperbolic differential equation of heat conduction of Lord and Shulman [9]

$$\rho C (\dot{T} + \tau_o \ddot{T}) = k T_{,ii} \tag{9}$$

where τ_o is a non-negative constant and is called the relaxation time.

According to Eq. (9), Kaliski [10] and Lord and Shulman [9] constructed the theory of generalized thermoelasticity.

Now, a new formula of heat conduction will be considered taking into account considerations (4), (5), and (8) as follows:

$$q_i + \tau_o \dot{q}_i = -k I^{\alpha-1} T_{,i}, \quad 0 < \alpha \leq 2 \tag{10}$$

where I is an integral operator, which is defined in Eq. (2).

In the context of the generalized thermoelasticity, the governing equations for isotropic medium are defined as follows.

For the equation of motion,

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$$\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i \tag{11}$$

For the constitution relation,

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij} \tag{12}$$

where λ and μ are Lamé's constant; u_i is the displacement component; F_i is the body force component; $\theta = |T - T_o|$ is the increment of the dynamical temperature, where T_o is the reference temperature; $\gamma = (3\lambda + 2\mu)\alpha_T$, where α_T is called the thermal expansion coefficient, where δ_{ij} is the Kronecker delta symbol; σ_{ij} is the stress tensor such that $\sigma_{ij} = \sigma_{ji}$; and e_{ij} is the strain tensor that satisfies the relations

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{and} \quad e = e_{11} + e_{22} + e_{33} \tag{13}$$

For the heat flux equation,

$$q_{i,i} = -\rho C\dot{\theta} - T_o \gamma \dot{e} \tag{14}$$

The entropy increment equation per unit volume takes the form

$$\rho T_o \eta = \rho C \theta + T_o \gamma e \tag{15}$$

where η is the entropy increment of the material.

For the heat flux-entropy equation,

$$q_{i,i} = -\rho T_o \dot{\eta} \tag{16}$$

For the heat equation without any heat sources

$$q_i + \tau_o \frac{\partial q_i}{\partial t} = -kI^{\alpha-1}\theta_{,i}, \quad 0 < \alpha \le 2 \tag{17}$$

By using Eqs. (15)–(17), we have the heat equation in the form

$$kI^{\alpha-1}\theta_{,ii} = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)(\rho C\theta + T_o \gamma e) \quad 0 < \alpha \le 2 \tag{18}$$

Where

$0 < \alpha < 1$ for weak conductivity

$\alpha = 1$ for normal conductivity

$1 < \alpha \le 2$ for strong conductivity

3 The Uniqueness Theorem

Let V be an open regular region of space with boundary S occupied by the reference configuration of a homogeneous isotropic linear thermoelastic solid. S is assumed closed and bounded. We supplement the equations of two temperature-generalized thermoelasticity (11)–(18) by prescribed boundary conditions as in Ref. [11]

$$u_i = \bar{u}_i \quad \text{on} \quad S_1 \times [0,\infty] \tag{19}$$

$$p_i = \bar{p}_i = \sigma_{ji} n_j \quad \text{on} \quad S - S_1 \times [0,\infty] \tag{20}$$

$$\theta_i = \bar{\theta}_i \quad \text{on} \quad S \tag{21}$$

where $S_1 \subset S$.

In addition, we have prescribed initial conditions

$$u_i = u_{i0}, \quad \dot{u}_i = \dot{u}_{i0}, \quad \theta = \theta_0 \quad \text{in} \quad V \quad \text{at} \quad t = 0 \tag{22}$$

4 Theorem

Given a regular region of space $V+S$ with volume V and boundary S, then there exists at most one set of single valued functions $\sigma_{ij}(x_k,t)$ and $e_{ij}(x_k,t)$ of class $C^{(1)}$, $u_i(x_k,t)$, and $\theta_i(x_k,t)$ of class $C^{(2)}$ in $V+S$, $t \ge 0$, which satisfy Eqs. (11)–(18) and conditions (19)–(22), where k, C, λ, μ, γ, T_o, ρ, and τ_o are all positive.

5 Proof

Let there be two sets of functions $\sigma_{ij}^{(\mathrm{I})}$ and $\sigma_{ij}^{(\mathrm{II})}$, $e_{ij}^{(\mathrm{I})}$ and $e_{ij}^{(\mathrm{II})}$, $\theta^{(\mathrm{I})}$ and $\theta^{(\mathrm{II})}\ldots$, etc. and let $\sigma_{ij} = \sigma_{ij}^{(\mathrm{I})} - \sigma_{ij}^{(\mathrm{II})}$, $e_{ij} = e_{ij}^{(\mathrm{I})} - e_{ij}^{(\mathrm{II})}$, $\theta = \theta^{(\mathrm{I})} - \theta^{(\mathrm{II})}\ldots$, etc.

By virtue of the linearity of the problem, it is clear that these differences also satisfy the above-mentioned equations (with $F_i = 0$) and homogeneous counterparts of conditions (19)–(22), namely, they satisfy the following field equations in $V \times (0,\infty)$:

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{23}$$

$$\sigma_{ij} = \sigma_{ji} \tag{24}$$

$$q_{i,i} = -\rho T_o \dot{\eta} \tag{25}$$

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij} \tag{26}$$

$$q_i + \tau_o \dot{q}_i = -kI^{\alpha-1}\theta_{,i}, \quad 0 < \alpha \le 2 \tag{27}$$

$$\rho T_o \eta = \rho C\theta + T_o \gamma_{ij} e_{ij} \tag{28}$$

$$kI^{\alpha-1}\theta_{,ii} = \rho C(\dot{\theta} + \tau_o \ddot{\theta}) + \gamma T_0(\dot{e}_{kk} + \tau_0 \ddot{e}_{kk}) \tag{29}$$

and

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{30}$$

together with the following boundary conditions:

$$u_i = 0 \quad \text{on} \quad S_1 \times [0,\infty) \tag{31}$$

$$p_i = \sigma_{ji} n_j = 0 \quad \text{on} \quad S - S_1 \times [0,\infty) \tag{32}$$

$$\theta_i = 0 \quad \text{on} \quad S \tag{33}$$

where $S_1 \subset S$.

In addition, we have the initial conditions

$$u_i = u_{i0}, \quad \dot{u}_i = \dot{u}_{i0}, \quad \theta = \theta_0 \quad \text{in} \quad V \quad \text{at} \quad t = 0 \tag{34}$$

Now, we will consider the integral

$$\int_v \sigma_{ij}\dot{e}_{ij}dv = \int_v \sigma_{ij}\dot{u}_{i,j}dv = -\int_v \sigma_{ij,j}\dot{u}_i dv \tag{35}$$

Upon inserting Eq. (23) the latter equation is reduced to

$$\int_v (\sigma_{ij}\dot{e}_{ij}dv + \rho \dot{u}_i \ddot{u}_i)dv = 0 \tag{36}$$

Using Eq. (26), we get

$$\int_v [(2\mu e_{ij} + \lambda \delta_{ij} e_{kk} - \gamma\theta\delta_{ij})\dot{e}_{ij} + \rho \dot{u}_i \ddot{u}_i]dv = 0 \tag{37}$$

This can be written as follows:

$$\frac{d}{dt}\int_v \left[\frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \frac{\rho \dot{u}_i \dot{u}_i}{2}\right]dv - \int_v \gamma\theta \dot{e}_{kk} dv = 0 \tag{38}$$

Substituting for $\dot{e}_{kk}$ in Eq. (29), we get

$$T_0 \frac{d}{dt}\int_v \left[\frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \frac{\rho \dot{u}_i \dot{u}_i}{2} + \frac{\rho C}{2T_0}\theta^2\right]dv - k\int_v \theta I^{\alpha-1}\theta_{,ii}dv$$

$$+ \tau_o \rho C \int_v \theta\ddot{\theta}dv + \gamma T_0 \tau_o \int_v \theta \ddot{e}_{kk} dv = 0 \tag{39}$$

where $\int_v I^{\alpha-1} f(v,t)dv = I^{\alpha-1}\int_v f(v,t)dv$ (v and t are independent variables).

Integrating by parts, we obtain

$$T_0\frac{d}{dt}\int_v\left[\frac{1}{2}\lambda e_{kk}^2+\mu e_{ij}e_{ij}+\frac{\rho\dot{u}_i\dot{u}_i}{2}+\frac{\rho C}{2T_0}\theta^2\right]dv+k\int_v\theta_{,i}I^{\alpha-1}\theta_{,i}dv$$

$$+\tau_o\rho C\int_v\theta\ddot{\theta}dv+\gamma T_0\tau_o\int_v\theta\ddot{e}_{kk}dv=0 \tag{40}$$

From the well-known inequality of the second law of thermodynamics

$$-q_i\theta_{,i}\geq 0 \tag{41}$$

By using Eq. (27), we get

$$k\int_v\theta_{,i}I^{\alpha-1}\theta_{,i}dv+\tau_o\int_v\dot{q}_i\theta_{,i}dv\geq 0$$

Integrating by parts, we obtain

$$k\int_v\theta_{,i}I^{\alpha-1}\theta_{,i}dv-\tau_o\int_v\dot{q}_{i,i}\theta dv\geq 0 \tag{42}$$

Inserting Eqs. (25) and (28) in the last equation, we get

$$k\int_v\theta_{,i}I^{\alpha-1}\theta_{,i}dv+\tau_o\rho C\int_v\theta\ddot{\theta}dv+\gamma T_o\tau_o\int_v\theta\ddot{e}_{kk}dv\geq 0 \tag{43}$$

Hence, we have

$$\frac{d}{dt}\int_v\left[\frac{1}{2}\lambda e_{kk}^2+\mu e_{ij}e_{ij}+\frac{\rho\dot{u}_i\dot{u}_i}{2}+\frac{\rho C}{2T_0}\theta^2\right]dv\leq 0 \tag{44}$$

The integral in the left hand side of Eq. (44) is initially zero since the difference functions satisfy homogeneous initial conditions. By inequality (44), however, this integral either decreases (or therefore becomes negative) or remains equal to zero. Since the integral is the sum of squares, only the latter alternative is possible, that is,

$$\int_v\left[\frac{1}{2}\lambda e_{kk}^2+\mu e_{ij}e_{ij}+\frac{\rho\dot{u}_i\dot{u}_i}{2}+\frac{\rho C}{2T_0}\theta^2\right]dv=0,\quad t\geq 0 \tag{45}$$

It follows that the difference functions are identically zero throughout the body and for all time this completes the proof of the theorem.

6 One-Dimensional Application

We will consider a half-space filled with an elastic material, which has constant elastic parameters. The governing equations will be written in the context of the fractional ordered generalized thermoelasticity theory.

The heat conduction equation takes the form

$$kI^{\alpha-1}\theta_{,ii}=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)(\rho C\theta+\gamma T_0 e),\quad 0<\alpha\leq 2 \tag{46}$$

The constitutive equation takes the form

$$\sigma_{ij}=2\mu e_{ij}+\lambda e_{kk}\delta_{ij}-\gamma\theta\delta_{ij} \tag{47}$$

The equation of motion without body force takes the form

$$\sigma_{ij,j}=\rho\ddot{u}_i \tag{48}$$

Now, we will suppose elastic and homogenous half-space $0\leq x<\infty$, which obey Eqs. (46)–(48) and initially quiescent, where all the state functions are dependent only on the dimension x and the time t.

The displacement components for the one-dimensional medium have the form

$$u_x=u(x,t),\quad u_y=u_z=0 \tag{49}$$

The strain component takes the form

$$e=e_{xx}=\frac{\partial u}{\partial x} \tag{50}$$

The heat conduction equation takes the form

$$kI^{\alpha-1}\frac{\partial^2\theta}{\partial x^2}=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)(\rho C\theta+\gamma T_0 e),\quad 0<\alpha\leq 2 \tag{51}$$

The constitutive equation takes the form

$$\sigma=\sigma_{xx}=(2\mu+\lambda)e-\gamma\theta \tag{52}$$

The equation of motion takes the form

$$\frac{\partial\sigma}{\partial x}=\rho\ddot{u} \tag{53}$$

or

$$\frac{\partial^2\sigma}{\partial x^2}=\rho\ddot{e} \tag{54}$$

For simplicity, we will use the following nondimensional variables:

$$x'=c_0\eta x,\quad \iota'=c_0^2\eta\iota,\quad \tau_0'=c_0^2\eta\tau_0,\quad \theta'=\frac{T-T_0}{T_0},\quad \sigma'=\frac{\sigma}{2\mu+\lambda} \tag{55}$$

where $c_0^2=[(2\mu+\lambda)/\rho]$ and $\eta=\rho C/k$.

Hence, we have (for simplicity, the primes have been dropped)

$$I^{\alpha-1}\frac{\partial^2\theta}{\partial x^2}=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)(\theta+\varepsilon e) \tag{56}$$

$$\sigma=e-\omega\theta \tag{57}$$

$$\frac{\partial^2\sigma}{\partial x^2}=\ddot{e}=\frac{\partial\ddot{u}}{\partial x} \tag{58}$$

where $\varepsilon=\gamma/\rho\mathrm{C}$ and $\omega=\gamma T_0/(\lambda+2\mu)$ are nondimensional constants.

Taking the Laplace transform for the both sides of Eqs. (56)–(58), this is defined as follows:

$$L\{f(t)\}=\bar{f}(s)=\int_0^\infty f(t)e^{-st}dt$$

we obtain

$$\frac{1}{s^{\alpha-1}}\frac{d^2\bar{\theta}}{dx^2}=(s+\tau_0s^2)\bar{\theta}+(s+\tau_0s^2)\varepsilon\bar{e} \tag{59}$$

where the rule for the Laplace transform of the Riemann–Liouville fractional integral reads from Ref. [4]

$$L\{I^nf(t)\}=\frac{1}{s^n}L\{f(t)\},\quad n>0 \tag{60}$$

$$\bar{\sigma}=\bar{e}-\omega\bar{\theta} \tag{61}$$

$$\frac{d^2\bar{\sigma}}{dx^2}=s^2\bar{e}=s^2\frac{d\bar{u}}{dx} \tag{62}$$

where all the initial state functions are equal to zero.

Eliminating $\bar{e}$ and $\bar{\theta}$ from Eqs. (59), (61), and (62), we obtain

$$\frac{d^2\bar{\theta}}{dx^2}=L_1\bar{\theta}+L_2\bar{\sigma} \tag{63}$$

where

$$L_1=(s^\alpha+\tau_0s^{\alpha+1})(1+\varepsilon\omega),\quad L_2=\varepsilon(s^\alpha+\tau_0s^{\alpha+1})$$

and

$$\frac{d^2\bar{\sigma}}{dx^2} = M_1\bar{\theta} + M_2\bar{\sigma} \tag{64}$$

where

$$M_1 = \omega s^2, \quad M_2 = s^2$$

Choosing as a state variable the temperature of heat conduction $\bar{\varphi}$ and the stress component $\bar{\sigma}$ in the x-direction, then Eqs. (63) and (64) can be written in matrix form, as in Ref. [12], as follows:

$$\frac{d^2\bar{V}(x,s)}{dx^2} = A(s)\bar{V}(x,s) \tag{65}$$

where

$$V(x,s) = \begin{bmatrix} \bar{\theta}(x,s) \\ \bar{\sigma}(x,s) \end{bmatrix} \quad \text{and} \quad A(s) = \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{bmatrix}$$

The formal solution of system (65) can be written in the form

$$\bar{V}(x,s) = \exp[-\sqrt{A(s)}x]\bar{V}(0,s) \tag{66}$$

where

$$V(0,s) = \begin{bmatrix} \bar{\theta}(0,s) \\ \bar{\sigma}(0,s) \end{bmatrix} = \begin{bmatrix} \bar{\theta}_0 \\ \bar{\sigma}_0 \end{bmatrix}$$

where for bounded solution with large x, we have canceled the part of the exponential that has a positive power.

We will use the well-known Cayley–Hamilton theorem to find the form of the matrix $\exp[-\sqrt{A(s)}x]$. The characteristic equation of the matrix $\sqrt{A(s)}$ can be written as follows:

$$k^2 - k(L_1 + M_2) + (L_1M_2 - L_2M_1) = 0 \tag{67}$$

the roots of this equation, namely, k_1 and k_2, satisfy the following relations:

$$k_1 + k_2 = L_1 + M_2 \tag{68}$$

$$k_1k_2 = L_1M_2 - L_2M_1 \tag{69}$$

Now, we can write the spectral decomposition of $A(s)$, as in Ref. [13]

$$A(s) = k_1E_1 + k_2E_2 \tag{70}$$

where E_1 and E_2 are called the projectors of $A(s)$ and satisfy the following conditions:

$$E_1 + E_2 = I \tag{71a}$$

$$E_1E_2 = \text{zero matrix} \tag{71b}$$

$$E_i^2 = E_i \quad \text{for} \quad i = 1,2 \tag{71c}$$

Then, we have

$$\sqrt{A(s)} = \sqrt{k_1}E_1 + \sqrt{k_2}E_2 \tag{72}$$

where

$$E_1 = \frac{1}{k_1 - k_2}\begin{bmatrix} (L_1 - k_2) & L_2 \\ \frac{(k_1 - L_1)(L_1 - k_2)}{L_2} & (k_1 - L_1) \end{bmatrix} \tag{73}$$

and

$$E_2 = \frac{1}{k_1 - k_2}\begin{bmatrix} (k_1 - L_1) & -L_2 \\ \frac{(k_1 - L_1)(k_2 - L_1)}{L_2} & (L_1 - k_2) \end{bmatrix} \tag{74}$$

Then, we get

$$B(s) = \sqrt{A(s)} = \frac{1}{\sqrt{k_1} + \sqrt{k_2}}\begin{bmatrix} \sqrt{k_1k_2} + L_1 & L_2 \\ M_1 & \sqrt{k_1k_2} + M_2 \end{bmatrix} \tag{75}$$

The Taylor series expansion for the matrix exponential in Eq. (66) is given by

$$\exp[-B(s)x] = \sum_{n=o}^{\infty} \frac{[-B(s)x]^n}{n!} \tag{76}$$

Using the Cayley–Hamilton theorem, we can express B^2 and higher orders of the matrix B in terms of I, B, where I is the unit matrix of second order.

Thus, the infinite series in Eq. (76) can be reduced to the following form:

$$\exp[-B(s)x] = b_0(x,s)I + b_1(x,s)B(s) \tag{77}$$

where b_0 and b_1 are coefficients depending on s and x.

Consider the characteristic roots p_1 and p_2 satisfy the characteristic equation of the matrix $B(s)$, which takes the form

$$P^2 - P(\sqrt{k_1} + \sqrt{k_2}) + \sqrt{k_1}\sqrt{k_2} = 0 \tag{78}$$

which gives

$$P_1 = \sqrt{k_1} \quad \text{and} \quad P_2 = \sqrt{k_2} \tag{79}$$

By the Cayley–Hamilton theorem, the roots of matrix B must satisfy Eq. (77); thus, we have

$$\exp(-p_1x) = b_0 + b_1p_1 \tag{80}$$

and

$$\exp(-p_2x) = b_0 + b_1p_2 \tag{81}$$

By solving the above linear system of equations, we get

$$b_0 = \frac{1}{p_1 - p_2}[p_1e^{-p_2x} - k_2e^{-p_1x}] \tag{82}$$

and

$$b_1 = \frac{1}{p_1 - p_2}[e^{-p_1x} - e^{-p_2x}] \tag{83}$$

Hence, we have

$$\exp[-B(s)x] = L_{ij}(x,s), \quad i,j = 1,2 \tag{84}$$

where

$$L_{11} = \frac{1}{k_1 - k_2}[e^{-\sqrt{k_2}x}(k_1 - L_1) - e^{-\sqrt{k_1}x}(k_2 - L_1)]$$

$$L_{12} = \frac{L_2}{k_1 - k_2}[e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}]$$

$$L_{22} = \frac{1}{k_2 - k_1}[e^{-\sqrt{k_1}x}(k_2 - M_2) - e^{-\sqrt{k_2}x}(k_1 - M_2)]$$

$$L_{21} = \frac{M_1}{k_1 - k_2}[e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}]$$

We can write the solution of Eq. (65) in the following form:

$$\bar{V}(x,s) = L_{ij}\bar{V}(0,s) \tag{85}$$

Hence, we obtain

$$\bar{\theta} = \frac{1}{k_1 - k_2}[(k_1\theta_0 - L_1\theta_0 - L_2\sigma_0)e^{-\sqrt{k_2}x} - (k_2\theta_0 - L_1\theta_0 - L_2\sigma_0)e^{-\sqrt{k_1}x}] \tag{86}$$

$$\bar{\sigma} = \frac{1}{k_1 - k_2}[(k_1\sigma_0 - M_1\theta_0 - M_2\sigma_0)e^{-\sqrt{k_2}x} - (k_2\sigma_0 - M_1\theta_0 - M_2\sigma_0)e^{-\sqrt{k_1}x}] \tag{87}$$

Now, we will use the boundary conditions on the boundary plane $x=0$, which is given by the following:

(1) Thermal boundary condition

We will suppose that, the boundary plane x=0 is subjected to a thermal shock as follows:

$$\theta(0,t) = \theta_0 = \theta_0^0 H(t) \tag{88}$$

where $H(t)$ is called the Heaviside unite step function and θ_0^0 is constant.

By using Laplace transform as we defined before, we get

$$\bar{\theta}(0,s) = \bar{\theta}_0 = \frac{\theta_0^0}{s} \tag{89}$$

(2) Mechanical boundary condition

We will consider that the boundary plane $x=0$ traction free, so we have

$$\sigma(0,t) = \sigma_0 = 0 \tag{90}$$

the above equation gives, after using the Laplace transform, the following equation:

$$\bar{\sigma}(0,t) = \bar{\sigma}_0 = 0 \tag{91}$$

Hence, we can use conditions (89) and (91) into the Eqs. (86) and (87) to get the solutions as follows:

$$\bar{\theta} = \frac{\theta_0^0}{s(k_1 - k_2)}[(k_1 - L_1)e^{-\sqrt{k_2}x} - (k_2 - L_1)e^{-\sqrt{k_1}x}] \tag{92}$$

$$\bar{\sigma} = \frac{\theta_0^0 M_1}{s(k_1 - k_2)}[e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}] \tag{93}$$

From Eq. (62), we have

$$\bar{u} = \frac{1}{s^2}\frac{d\bar{\sigma}}{dx} \tag{94}$$

Substituting from Eq. (85) into Eq. (86), we get

$$\bar{u} = \frac{-\theta_0^0 M_1}{s^3(k_1 - k_2)}[\sqrt{k_1}e^{-\sqrt{k_1}x} - \sqrt{k_2}e^{-\sqrt{k_2}x}] \tag{95}$$

Those complete the solution in the Laplace transform domain.

7 Inverse Laplace Transforms

In order to invert the Laplace transforms, we adopt a numerical inversion method based on a Fourier series expansion in Ref. [14].

By this method the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t) = \frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{f}(c) + R1\sum_{k=1}^{N}\bar{f}\left(c + \frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right], \quad 0 < t_1 < 2t \tag{96}$$

where N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$\exp(ct)R1\left[\bar{f}\left(c + \frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right] \le \varepsilon_1 \tag{97}$$

where ε_1 is a prescribed small positive number that corresponds to the degree of accuracy required. Parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of c was obtained according to the criteria described in Ref. [14].

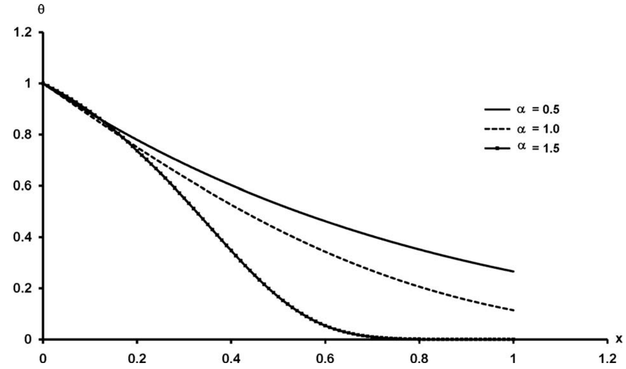

Fig. 1 The temperature distribution at $t=0.2$

8 Numerical Results and Discussion

The copper material was chosen for purposes of numerical evaluations, and the constants of the problem were taken from Refs. [12,15] as follows:

$$k = 386 \text{ N/K s}, \quad \alpha_T = 1.78 \times 10^{-5} \text{ K}^{-1}, \quad C = 383.1 \text{ m}^2/\text{K}$$

$$\eta = 8886.73 \text{ m/s}^2, \quad \mu = 3.86 \times 10^{-10} \text{ N/m}^2$$

$$\lambda = 7.76 \times 10^{-10} \text{ N/m}^2, \quad \rho = 8954 \text{ kg/m}^3, \quad \tau_o = 0.02 \text{ s}$$

$$T_o = 293 \text{ K}, \quad \varepsilon = 1.60861, \quad \omega = 0.0104, \quad \theta_0^0 = 1$$

Figures 1–3 display the temperature distribution q, the stress distribution s, and the displacement distribution u for wide range of x

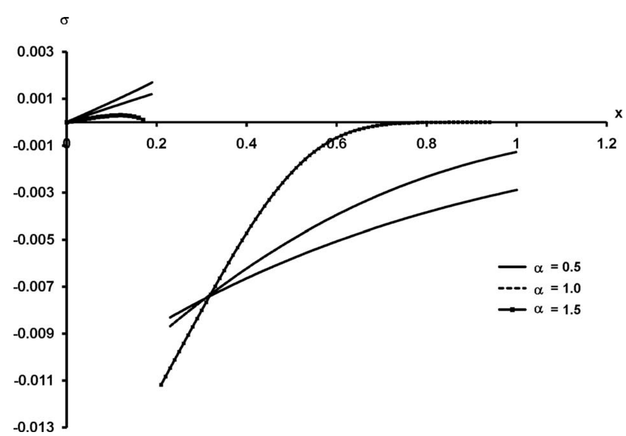

Fig. 2 The stress distribution at $t=0.2$

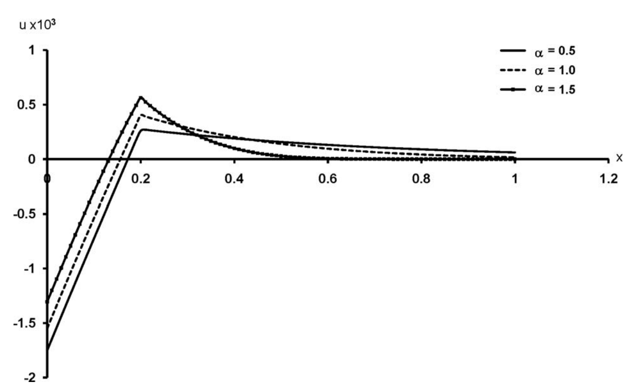

Fig. 3 The displacement distribution at $t=0.2$

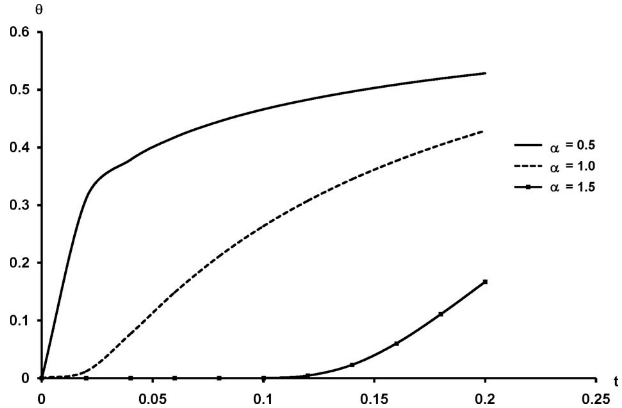

Fig. 4 The temperature distribution at $x=0.5$

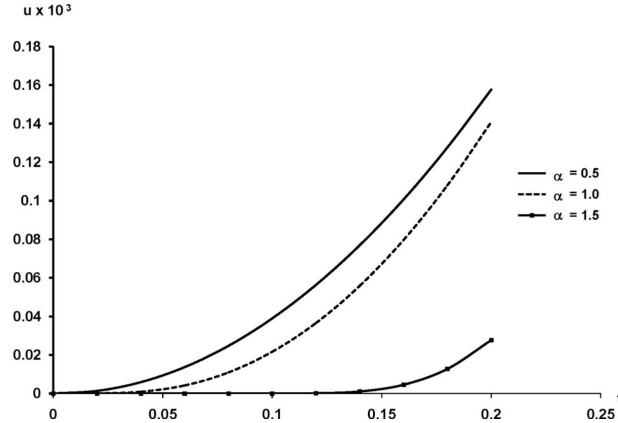

Fig. 6 The displacement distribution at $x=0.5$

$(0<x<1)$ at a small value of time $t=0.2$, and it is noticed that α has a significant effect on all the fields.

In Fig. 1, for a weak conductivity $0<\alpha=0.5<1$, the particle transports the heat to the other particle with difficulty, which makes the particles keep the temperature within itself for a longer time interval, which makes this curve lie above the other two curves. For a normal conductivity $\alpha=1.0$, the results coincide with all the previous results of applications that are taken in the context of the generalized thermoelasticity, as in Refs. [9,12,15]. For strong conductivity (superconductivity) $1<\alpha=1.5\leq 2$, the particles transport the heat to the other particles easily and this makes the decreasing rate of the temperature greater than the other ones.

In Fig. 2, the stress field has the same behavior as the temperature except at discontinuous points. To explain that discontinuous points, the effect of the thermal shock on the boundary generates a wave for small interval $0\leq\alpha\leq 0.19$; after this another wave is generated as a reaction of the first wave in inverse direction.

In Fig. 3, the displacement distribution has been affected by the value of α, where the maximum point of the displacement increases when the value of increases. During the same interval $0\leq\alpha\leq 0.19$, all the displacements increase with a very great rate until the maximum point at $x=0.19$.

Figures 4–6 display the temperature distribution q, the stress distribution s, and the displacement distribution u for wide range of time t $(0\leq t\leq 0.2)$ at $x=0.5$, and it is noticed that α has significant effects on all the fields. It is noticed that all the waves reach the steady state depending on the value of α.

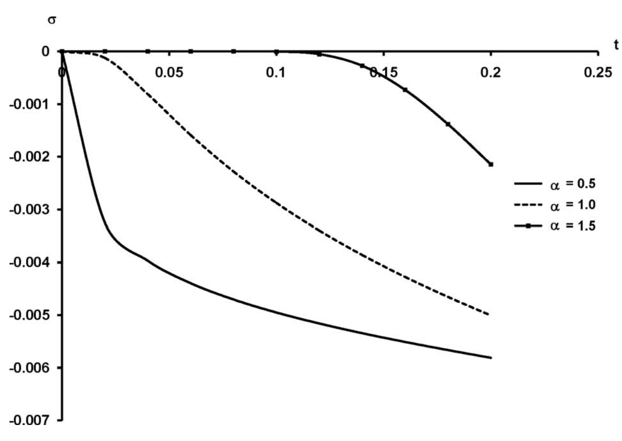

Fig. 5 The stress distribution at $x=0.5$

9 Conclusion

In general, we have the following system of equations that cover four theorems.

For the equation of motion,

$$\rho\frac{\partial^2 u_i}{\partial t^2}=\rho F_i+(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}-\gamma\left(\theta+\nu\frac{\partial\theta}{\partial t}\right)_{,i} \tag{98}$$

For the generalized heat conduction equation,

$$kI^{\alpha-1}\theta_{,ii}=\rho C\left(\frac{\partial\theta}{\partial t}+\tau_o\frac{\partial^2\theta}{\partial t^2}\right)+\gamma T_o\left(\frac{\partial u_{i,i}}{\partial t}+n_o\tau_o\frac{\partial^2 u_{i,i}}{\partial t^2}\right) \tag{99}$$

where

$$\begin{aligned}&0<\alpha<1 && \text{for weak conductivity}\\ &\alpha=1 && \text{for normal conductivity}\\ &1<\alpha\leq 2 && \text{for superconductivity}\end{aligned}$$

and n_o is a constant parameter.

For the constitutive equation,

$$\sigma_{ij}=\lambda e_{kk}\delta_{ij}+2\mu e_{ij}-\gamma\left(\theta+\nu\frac{\partial\theta}{\partial t}\right) \tag{100}$$

$$e_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{101}$$

The previous equations constitute a complete system of fractional order generalized thermoelasticity. This model can be applied to both classical generalizations, the Lord–Shulman theory ($n_0=1$, $\tau_0>0$, $\nu=0$, $\alpha=1$) and the Green–Lindsay theory ($n_0=0$, $\tau_0>0$, $\nu>0$, $\alpha=1$), as well as to the coupled theory ($\tau_0=\nu=0$, $\alpha=1$).

Nomenclature

- ρ = the mass density
- c = the concentration
- κ = the diffusion conductivity
- C = the specific heat
- k = the thermal conductivity
- q_i = the heat flux components
- T = the temperature
- τ_o = the relaxation time
- λ,μ = Lame's constants
- u_i = the displacement component
- F_i = the body force component
- θ = $|T-T_o|$ is the increment of the dynamical temperature
- T_o = the reference temperature
- α_T = the thermal expansion coefficient

δ_{ij} = the Kronecker delta symbol
σ_{ij} = the stress tensor such that
e_{ij} = the strain tensor
η = the entropy increment
$c_0^2 = (2\mu+\lambda)/\rho$
$\varepsilon = \gamma/\rho C$
$\omega = \gamma T_0/(\lambda+2\mu)$

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