background information for exc set 2 task 8
task 8. (Plane Curves: TheParametric Description)
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Confusing the notation : don't now how to handle vector-valued function **r** in Maple
A parameterized Curve can alternatively thought as a vector-valued function
To choose a curve in task 8 (i)(ii)(iii)

countered are the parabola $y - x^2 = 0$, the hyperbola $x^2 - y^2 - 1 = 0$, and the ellipse $x^2/4 + y^2 - 1 = 0$.

You will be assigned one of the following curves to study. For your assigned curve

(a) $H(x,y) = x^3 - xy^2 + 1$
(b) $H(x,y) = x^3 - 2y^2 + \frac{3}{2}$
(c) $H(x,y) = x^3 + y^2 - 2xy - 1$
(d) $H(x,y) = x^3 - x - y^2 + 1$
(e) $H(x,y) = x^3y - x - y^2 + 1$
(f) $H(x,y) = x^3y - x - xy^2 + 1$
(g) $H(x,y) = x^3 - y - x^2y^2 + 1$
(h) $H(x,y) = x^3y - x - x^2y^2 + 1$
(i) $H(x,y) = x^3y - x - x^2y^2 - y^2 + 1$
(j) $H(x,y) = x^2y + x^3 + xy^3 + y + 1$

do the following

(i) Use the `implicitplot` command to plot the portion of the curve that lies in the rectangle $[-4, 2] \times [-4, 5]$ (see Maple/Calculus Notes). Use a fine enough grid so that the curve appears smooth. In the same picture plot the circle $x^2 + y^2 = 1$ and use the `fsolve` command to find all points of intersection of this circle with the assigned curve. Annotate your picture, identifying the respective curves.

(ii) For $h = -.5, 0, .5$, use the `fsolve` command to find all solutions to $H(1 + h, y) = 0$. This will give one or more points $(1 + h, y)$ on the curve. Plot the tangent lines at these points (with suitable length) and the curve itself (for $x \in [0, 2]$) in the same picture. Use colors to distinguish the respective plots. Print out and annotate your picture.

(iii) Define two 1-dimensional arrays `p[i],c[i],i=1..10`. Store your favorite 10 colors in the array `c` (such as `c[2]:=magenta`). Then use a do loop to store the plots of the curves $H(x,y) - i/10 = 0$, for $i = 1, \ldots, 10$, in the array `p`, with `p[i]` in color `c[i]`. Display all the plots in the same picture. Annotate this picture to clearly exhibit which curve goes with which value of i.

8. **(Plane Curves: The Parametric Description)** If two functions $f, g : [a, b] \to \mathbb{R}$ are given on some interval $[a, b]$, then a curve is given parametrically by the two equations:

$$\begin{aligned} x &= f(t) \\ y &= g(t). \end{aligned}$$

Technically the curve is the set of points $C = \{(f(t), g(t)) \mid t \in [a, b]\}$. The functions f and g are called the x and y coordinate functions, respectively. The independent variable t is called the parameter and often represents time. For each $t \in [a, b]$ there is a corresponding point in the plane with coordinates $(f(t), g(t))$. Plotting these points as t varies over $[a, b]$ gives a collection of points in the plane which constitutes the curve. When t represents the time, the point $(f(t), g(t))$ represents the position of a particle moving in the plane and as t varies from a to b, the particle traces out the curve, which is also known as the trajectory of the particle. Note that in this sense, a parameterized curve has a direction associated to it, i.e., the direction in which the particle moves on its trajectory.

Also recall from calculus that a parameterized curve can alternatively be thought of as a vector-valued function $\mathbf{r} : [a, b] \to \mathbb{R}^2$, expressed by the formula

$$\mathbf{r}(t) = [f(t), g(t)],$$

for $t \in [a, b]$. Different calculus texts will use different notation for the component expressions for vectors. Here we use square brackets: $\mathbf{v} = [v_1, v_2]$, since this is how Maple represents vectors, namely, as lists, or 1-dimensional arrays. From a physical point of view, $\mathbf{r}(t)$ is the position vector of the particle at time t. While a vector can be plotted anywhere in the plane, the position vector is usually plotted with its initial point at the origin. Then its terminal point is $(f(t), g(t))$.

The first and second derivatives of the vector-valued function $\mathbf{r}$ are also vector-valued functions and in physics are called the velocity and acceleration function for the moving particle:

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = [f'(t), g'(t)] \\ \mathbf{a}(t) &= \mathbf{r}''(t) = [f''(t), g''(t)]. \end{aligned}$$

The vector $\mathbf{v}(t)$, when plotted with its initial point at $(f(t), g(t))$ is tangent to the curve. The vector $\mathbf{a}(t)$, when plotted with its initial point at $(f(t), g(t))$ should point in the direction in which the curve is curving.

You will be assigned one of the following curves to study:

(a) $\mathbf{r}(t) = [t^4 - t, 2t^3 - t^5]$, for $t \in [-1.55, 1.55]$.

(b) $\mathbf{r}(t) = [t^5 - t^2 + 1, 2t^3 - t^2 - t + 1]$, for $t \in [-0.8, 1.2]$.

(c) $\mathbf{r}(t) = [t^6 - t^2 + 1, t^3 - t^2 - t + 1]$, for $t \in [-1.1, 1.25]$.

(d) $\mathbf{r}(t) = [t^4 - t^2 + 1, 2t^2 - t + 1]$, for $t \in [-0.8..1.2]$.

(e) $\mathbf{r}(t) = [t^4 - t^3 + 1, 2t^3 - t^2 - t + 1]$, for $t \in [-0.8, 1.2]$.

(f) $\mathbf{r}(t) = [t^4 - t^3 + 1, 2t^4 - t^2 - t + 1]$, for $t \in [-0.8, 1.2]$.

(g) $\mathbf{r}(t) = [t^5 - t^2 + 1, 2t^4 - t^2 - t + 1]$, for $t \in [-0.8, 1.2]$.

(h) $\mathbf{r}(t) = [t^5 - t^2 + 1, 2t^5 - t^2 - t + 1]$, for $t \in [-0.8, 1.2]$.

(i) $\mathbf{r}(t) = [t^5 - t + 1, 2t^6 - t^2 - t + 1]$, for $t \in [-1, 1.2]$.

For your particular curve do the following:

(i) Study the way the velocity and acceleration change as the curve is swept out. Do this by choosing a point $Q = \mathbf{r}(t_0)$ on the curve (your choice, but choose an "interesting" one) and then plotting the tangent (or velocity) lines and acceleration lines at the points $Q_i = \mathbf{r}(t_0 + i\Delta t)$, for $i = 0, 1, 2, 3$. For this choose a suitably small Δt so that the points Q_i are not too far apart, yet far enough apart to be distinguished from each other. Store the plots of the velocity and acceleration lines in two arrays `vel` and `accel`. For the velocity lines, you will have to experiment in order to choose a suitable length, but plot each as a line segment starting at its respective Q_i and extending in the "forward" direction. Use an array of colors `c[i],i=0..3` to render each velocity line in a different color. Do a similar thing for the acceleration lines. Plot all of these and the curve itself in the same picture. Mark the directions on the lines and curve and annotate the figure (by hand after printing out).

(ii) Use `fsolve` to find all points of intersection of your curve with itself. Print out and annotate a figure with this information.

(iii) Use `fsolve` to find all points of intersection of your curve with the curve:

$$\mathbf{r}(t) = [t^3 - t, t^4 - t^2],$$

for $t \in [a, b]$. You will have to select a suitable interval $[a, b]$ by experimentation. Plot both curves in the same picture and mark your answer on the printout of the picture.

2.8 Maple/Calculus Notes

We review here a number of topics from both calculus and Maple for dealing with plane curves. Recall that a plane curve can be represented in two ways: by an equation $H(x, y) = 0$ (the implicit description) and parametrically by a pair of equations involving a single parameter (the parametric description).

2.8.1 Planes Curves Given Implicitly

(1) To define a function $H(x, y) = xy^2 - y^3 + 1$ of two variables in Maple use the arrow notation much as for a function of one variable:

```
H:=(x,y)->x*y^2-y^3+1;
```

To plot the curve with equation $H(x, y) = 0$, use the `implicitplot` command. This command is part of the `plots` package so you have to load this package into memory first.

```
with(plots):
implicitplot(H(x,y)=0,x=a..b,y=c..d,color=blue);
```

If $G(x, y) = 0$ is another curve, you can plot both curves together with the command

```
implicitplot({H(x,y)=0,G(x,y)=0},x=a..b,y=c..d,color=blue);
```

If you want to use different ranges for x and y or to use different options (such as color, grid size, etc.) for the curves, you will have to plot each curve separately and then display them in the same picture:

```
p:=implicitplot(H(x,y)=0,x=a..b,y=c..d,color=blue):
q:=implicitplot(G(x,y)=0,x=e..f,y=r..s,color=black,
               grid=[n,k]):
display({p,q});
```

The `grid=[n,k]` option causes Maple to apply its plotting routine with a grid of points obtained by subdividing $[e, f]$ into $n-1$ equal subintervals and $[r, s]$ into $k-1$ subintervals to produce a grid of nk points. The default is a 25×25 grid of 625 points.

(2) You can use the `fsolve` command to solve systems of equations, much as you have used it to solve a single equation. There must be as many equations as there are unknowns. For example, suppose there are two equations with two unknowns:

$$\begin{aligned} H(x, y) &= 0 \\ G(x, y) &= 0 \end{aligned}$$

Assuming the functions H and G have been previously been defined in your Maple session, you can find a solution in the rectangle $[a, b] \times [c, d]$ with the command

```
fsolve({H(x,y)=0,G(x,y)=0},{x,y},x=a..b,y=c..d);
```


Of course there may not be a solution in this rectangle, in which case the command is just echoed back as output. Geometrically, the solutions (x, y) of the system give the points of intersection of the two curves with equations $H(x, y) = 0$ and $G(x, y) = 0$.

(3) To calculate the slope of the tangent line to the curve $H(x, y) = 0$ at a point $P = (x_0, y_0)$, you need to use implicit differentiation and find the derivative of y with respect to x. In Maple use the `implicitdiff` command,

```
implicitdiff(H(x,y),y,x);
```

to get the formula for dy/dx . Here as in the `diff` command the result is an expression, not a function of x and y. To convert it to a function use the `unapply` command.

```
M:=unapply(implicitdiff(H(x,y),y,x),(x,y));
```

Now you can evaluate M at the point P to get the number $M(x_0, y_0)$, which is the slope of the tangent line at P.

```
plot({[f(t),g(t),t=a..b],[h(s),k(s),s=c..d]});
```

Of course you can use the same parameter name `t` instead of `s` in doing the second plot if you wish. To have different colors (or other options) for each curve, you will have to create separate plot structures and then use the `display` command.

(2) If $\mathbf{r}(t) = [f(t), g(t)]$ and $\mathbf{q}(s) = [h(s), k(s)]$, for $t \in [a, b]$ and $s \in [c, d]$ are two parameterized plane curves, then the points where these curves intersect can be found by solving a system of equations. To see this, note that if the curves intersect at a point $P = (x_0, y_0)$, then there are values t, s of the respective parameters for which $\mathbf{r}(t) = P$ and $\mathbf{q}(s) = P$. Thus, we get the vector equation

$$\mathbf{r}(t) = \mathbf{q}(s),$$

which, when written out fully, gives a pair of two equations

$$\begin{aligned} f(t) &= h(s) \\ g(t) &= k(s) \end{aligned}$$

for the two unknowns t, s. If we can find a solution of this system, i.e., a t and s (which generally will be different numbers), then we can use either t or s to compute P.