After substituting $f_{1,n}(\xi)$ and $f_{2,n}(\xi)$ into (6.24) and (6.25), and subsequently substituting $u_{1,n}(x)$ and $u_{2,n}(x)$ into the first series in (6.11), we finally obtain the solution of the boundary-value problem of (6.7)–(6.10) in integral form

$$u_1(x,y) = \int_0^b \int_{-\infty}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{11}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$

$$+ \int_0^b \int_0^{\infty} \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{12}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_1, \quad (6.26)$$

and

$$u_2(x,y) = \int_0^b \int_{-\infty}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{21}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$

$$+ \int_0^b \int_0^{\infty} \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{22}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_2. \quad (6.27)$$

Upon inspection of above integrals and recalling the relation in (6.6), where $u_i(x,y)$ and $f_i(x,y)$, with $i = 1,2$ in (6.26) and (6.27), defining the components of the vector-functions $\mathbf{U}(x,y)$ and $\mathbf{F}(x,y)$

$$U_i(x,y) = \begin{cases} u_i(x,y), & (x,y) \in \Omega_i, \\ 0, & (x,y) \notin \Omega_i, \end{cases} \quad i = 1,2,$$

and

$$F_i(x,y) = \begin{cases} f_i(x,y), & (x,y) \in \Omega_i, \\ 0, & (x,y) \notin \Omega_i, \end{cases} \quad i = 1,2,$$

we recognize that the kernel-functions

$$\frac{2}{b} \sum_{n=1}^{\infty} g_{ij}^n(x,\xi) \sin \nu y \sin \nu \eta, \quad i,j = 1,2, \quad (6.28)$$

in (6.26) and (6.27) represent the elements $G_{ij}(x,y;\xi,\eta)$ in the matrix of Green's type $\mathbf{G}(x,y;\xi,\eta)$ for the homogeneous boundary-value problem corresponding to (6.7)–(6.10).

Upon close analysis we find that the series expansions in (6.28), with the expressions for $g_{ij}^n(x,\xi)$ found earlier in this section, can be readily summed up with the aid of the standard summation formula

$$\sum_{n=1}^{\infty} \frac{q^n}{n} \cos n\alpha = -\frac{1}{2} \ln(1 - 2q\cos\alpha + q^2) \quad (6.29)$$

valid for $q \le 1$ and $0 \le \alpha < 2\pi$, and was utilized repeatedly, earlier in our book. Indeed, the expansion in (6.28) can be rewritten as

$$G_{ij}(x, y; \xi, \eta) = \frac{1}{b} \sum_{n=1}^{\infty} g_{ij}^{n}(x, \xi) \left[\cos \nu(y - \eta) - \cos \nu(y + \eta)\right], \quad i, j = 1, 2,$$

and the summation ultimately yields the following compact closed formulas

$$G_{11}(z, \zeta) = \frac{1}{2\pi} \left[\ln \frac{|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|} - \frac{1-\lambda}{1+\lambda} \ln \frac{|1 - e^{\omega(z+\overline{\zeta})}|}{|1 - e^{\omega(z+\zeta)}|} \right],$$

$$G_{12}(z, \zeta) = \frac{\lambda}{\pi(1+\lambda)} \ln \frac{|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|},$$

$$G_{21}(z, \zeta) = \frac{1}{\pi(1+\lambda)} \ln \frac{|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|}$$

and

$$G_{11}(z, \zeta) = \frac{1}{2\pi} \left[\ln \frac{|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|} + \frac{1-\lambda}{1+\lambda} \ln \frac{|1 - e^{\omega(z+\overline{\zeta})}|}{|1 - e^{\omega(z+\zeta)}|} \right]$$

for the elements of the matrix of Green's type $\mathbf{G}(x, y; \xi, \eta)$ for the homogeneous boundary-value problem, corresponding to that in (6.7)–(6.10). Here $\omega = \pi/b$. We introduced complex variable notations $z = x + iy$ and $\zeta = \xi + i\eta$ for the field point (x, y) and the source point (ξ, η), respectively, with the bar on ζ denoting its complex conjugate.

Clearly, the above expressions for the elements of the matrix of Green's type are computable immediately, since they represent real-valued functions, whilst the complex variables are only used for compactness.

It can be readily seen that, for λ equal to unity (that is, the materials occupying the fragments Ω_1 and Ω_2 of Ω are identical), the above expressions reduce to the well-known closed form (refer to, for example, Chapter 2)

$$G(z, \zeta) = \frac{1}{2\pi} \ln \frac{|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|}$$

of the Green's function for the Dirichlet problem for the Laplace equation in the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$.

Example 6.2. We continue with a mixed boundary-value problem for two Klein–Gordon equations stated on the segments $\Omega_1 = \{-a < x < 0, 0 < y < b\}$ and $\Omega_2 = \{0 < x < \infty, 0 < y < b\}$ of the compound semi-infinite strip $\Omega = \{-a <$

$x < \infty, 0 < y < b\}$. Let Ω_1 and Ω_2 be filled with materials whose conductivities are defined by λ_1 and λ_2, respectively. Consider now the problem

$$\nabla^2 u_i(x,y) - k_i^2 u_i(x,y) = -f_i(x,y), \quad (x,y) \in \Omega_i,\ i = 1,2, \tag{6.30}$$

$$\frac{\partial u_1(-a,y)}{\partial x} - \beta u_1(-a,y) = 0, \quad \lim_{x\to\infty} u_2(x,y) < \infty, \tag{6.31}$$

$$u_i(x,0) = 0, \quad \frac{\partial u_i(x,b)}{\partial y} = 0, \quad i = 1,2, \tag{6.32}$$

and

$$u_1(0,y) = u_2(0,y), \quad \frac{\partial u_1(0,y)}{\partial x} = \lambda \frac{\partial u_2(0,y)}{\partial x}, \tag{6.33}$$

where $\beta \geq 0$ and $\lambda = \lambda_2/\lambda_1$.

Due to the form of the conditions in (6.32), we can expand the unknown functions $u_i(x,y)$ and the right-hand side terms $f_i(x,y)$, $i = 1,2$, from the above equation into the Fourier series

$$u_i(x,y) = \sum_{n=1}^{\infty} u_{i,n}(x) \sin \nu y, \quad \nu = \frac{(2n-1)\pi}{2b}, \tag{6.34}$$

and

$$f_i(x,y) = \sum_{n=1}^{\infty} f_{i,n}(x) \sin \nu y. \tag{6.35}$$

After substituting these expansions in the original formulation of (6.30)–(6.33), we obtain the following set ($n = 1,2,3,\ldots$) of three-point-posed boundary-value problems

$$\frac{d^2 u_{1,n}(x)}{dx^2} - (\nu^2 + k_1^2) u_{1,n}(x) = -f_{1,n}(x), \quad x \in (-a,0), \tag{6.36}$$

$$\frac{d^2 u_{2,n}(x)}{dx^2} - (\nu^2 + k_2^2) u_{2,n}(x) = -f_{2,n}(x), \quad x \in (0,\infty), \tag{6.37}$$

$$\frac{du_{1,n}(-a)}{dx} - \beta u_{1,n}(-a) = 0, \quad \lim_{x\to\infty} u_{2,n}(x) < \infty, \tag{6.38}$$

$$u_{1,n}(0) = u_{2,n}(0), \quad \frac{du_{1,n}(0)}{dx} = \lambda \frac{du_{2,n}(0)}{dx} \tag{6.39}$$

for the coefficients $u_{1,n}(x)$ and $u_{2,n}(x)$ of the expansion in (6.34).

Clearly, fundamental sets of solutions for the homogeneous equations corresponding to (6.36) and (6.37) can be composed of, for example, the functions

$$e^{h_i x} \quad \text{and} \quad e^{-h_i x}$$

with $h_i = \sqrt{\nu^2 + k_i^2}$, for $i = 1,2$.

Hence, in accordance with the standard procedure of the method of variation of parameters, the solution of the boundary-value problem in (6.36)–(6.39) can be written in the form

$$u_{i,n}(x) = C_{i,n}(x)e^{h_i x} + D_{i,n}(x)e^{-h_i x}, \quad i = 1, 2, \tag{6.40}$$

where the coefficients $C_{i,n}(x)$ and $D_{i,n}(x)$ are to be determined. This yields the following well-posed system of linear algebraic equations

$$\begin{pmatrix} e^{h_i x} & e^{-h_i x} \\ h_i e^{h_i x} & -h_i e^{-h_i x} \end{pmatrix} \begin{pmatrix} C'_{i,n}(x) \\ D'_{i,n}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f_{i,n}(x) \end{pmatrix}, \quad i = 1, 2,$$

for the derivatives of $C_{i,n}(x)$ and $D_{i,n}(x)$ of (6.40). After solving the above system, we obtain

$$C'_{i,n}(x) = -\frac{1}{2h_i}e^{-h_i x} f_{i,n}(x), \quad D'_{i,n}(x) = \frac{1}{2h_i}e^{h_i x} f_{i,n}(x), \quad i = 1, 2.$$

Straightforward integration yields

$$C_{1,n}(x) = -\frac{1}{2h_1}\int_{-a}^{x} e^{-h_1\xi} f_{1,n}(\xi)d\xi + \gamma_1, \tag{6.41}$$

$$D_{1,n}(x) = \frac{1}{2h_1}\int_{-a}^{x} e^{h_1\xi} f_{1,n}(\xi)d\xi + \delta_1, \tag{6.42}$$

$$C_{2,n}(x) = -\frac{1}{2h_2}\int_{0}^{x} e^{-h_2\xi} f_{2,n}(\xi)d\xi + \gamma_2 \tag{6.43}$$

and

$$D_{2,n}(x) = \frac{1}{2h_2}\int_{0}^{x} e^{h_2\xi} f_{2,n}(\xi)d\xi + \delta_2. \tag{6.44}$$

Substituting these expressions in (6.40) and rearranging the integral terms appropriately, we obtain general solutions to the equations in (6.36) and (6.37) as

$$u_{1,n}(x) = \gamma_1 e^{h_1 x} + \delta_1 e^{-h_1 x} + \frac{1}{2h_1}\int_{-a}^{x} [e^{-h_1(x-\xi)} - e^{h_1(x-\xi)}] f_{1,n}(\xi)d\xi \tag{6.45}$$

and

$$u_{2,n}(x) = \gamma_2 e^{h_2 x} + \delta_2 e^{-h_2 x} + \frac{1}{2h_2}\int_{0}^{x} [e^{-h_2(x-\xi)} - e^{h_2(x-\xi)}] f_{2,n}(\xi)d\xi. \tag{6.46}$$

The constant coefficients $\gamma_1, \gamma_2, \delta_1$, and δ_2 in $u_{1,n}(x)$ and $u_{2,n}(x)$ as written above, can be obtained by imposing the boundary and contact conditions in (6.38) and (6.39). The boundedness of $u_{2,n}(x)$ as x goes to infinity allows one to directly find γ_2 (see the second condition in (6.38)). In order to do this, it is convenient to transform the expression for $u_{2,n}(x)$ in (6.46) into the equivalent form

$$u_{2,n}(x) = \left[-\frac{1}{2h_2}\int_0^x \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi + \gamma_2\right]\mathrm{e}^{h_2 x} + \left[\frac{1}{2h_2}\int_0^x \mathrm{e}^{h_2\xi} f_{2,n}(\xi)d\xi + \delta_2\right]\mathrm{e}^{-h_2 x}$$

from which it follows immediately that the factor of $\mathrm{e}^{h_2 x}$ must be zero, since $u_{2,n}(x)$ is required to be bounded as x goes to infinity. This yields

$$\gamma_2 = \frac{1}{2h_2}\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi.$$

The remaining conditions in (6.38) and (6.39) lead us to the well-posed system of linear algebraic equations

$$\begin{pmatrix} (h_1-\beta)\mathrm{e}^{-h_1 a} & -(h_1+\beta)\mathrm{e}^{h_1 a} & 0 \\ 1 & 1 & -1 \\ h_1 & -h_1 & \lambda h_2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ M \\ N \end{pmatrix} \tag{6.47}$$

in γ_1, δ_1,and δ_2, with M and N in the right-hand side vector defined as

$$M = \frac{1}{2h_1}\int_{-a}^0 (\mathrm{e}^{-h_1\xi} - \mathrm{e}^{h_1\xi}) f_{1,n}(\xi)d\xi + \frac{1}{2h_2}\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi,$$

$$N = \frac{1}{2}\int_{-a}^0 (\mathrm{e}^{h_1\xi} + \mathrm{e}^{-h_1\xi}) f_{1,n}(\xi)d\xi + \frac{\lambda}{2}\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi.$$

The solution of the system in (6.47) now is found as

$$\gamma_1 = \frac{h_1+\beta}{2h_1\Delta}\left\{\int_{-a}^0 [(h_1+\lambda h_2)\mathrm{e}^{-h_1\xi} + (h_1-\lambda h_2)\mathrm{e}^{h_1\xi}] f_{1,n}(\xi)d\xi + 2h_1\lambda\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi\right\}\mathrm{e}^{2h_1 a},$$

$$\delta_1 = \frac{h_1-\beta}{2h_1\Delta}\left\{\int_{-a}^0 [(h_1+\lambda h_2)\mathrm{e}^{-h_1\xi} + (h_1-\lambda h_2)\mathrm{e}^{h_1\xi}] f_{1,n}(\xi)d\xi + 2h_1\lambda\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi\right\}$$

and

$$\delta_2 = \frac{1}{\Delta}\left\{\int_{-a}^0 [(h_1-\beta)\mathrm{e}^{h_1\xi} + (h_1+\beta)\mathrm{e}^{h_1(2a+\xi)}] f_{1,n}(\xi)d\xi - \int_0^\infty \frac{h_1-\lambda h_2}{2h_2}[(h_1+\beta)\mathrm{e}^{2h_1 a - h_2\xi} + (h_1-\beta)\mathrm{e}^{-h_2\xi}] f_{2,n}(\xi)d\xi\right\},$$

where

$$\Delta = (h_1 + \lambda h_2)(h_1 + \beta)e^{2h_1 a} - (h_1 - \lambda h_2)(h_1 - \beta).$$

Upon substituting $\gamma_1, \gamma_2, \delta_1$, and δ_2 in (6.41)–(6.44), the functions $u_{1,n}(x)$ and $u_{2,n}(x)$ can be rewritten in compact form as

$$u_{1,n}(x) = \int_{-a}^{0} g_{11}^n(x,\xi) f_{1,n}(\xi) d\xi + \int_{0}^{\infty} g_{12}^n(x,\xi) f_{2,n}(\xi) d\xi \tag{6.48}$$

and

$$u_{2,n}(x) = \int_{-a}^{0} g_{21}^n(x,\xi) f_{1,n}(\xi) d\xi + \int_{0}^{\infty} g_{22}^n(x,\xi) f_{2,n}(\xi) d\xi \tag{6.49}$$

with the kernel-functions $g_{ij}^n(x,\xi)$ written as

$$\begin{aligned} g_{11}^n(x,\xi) &= \frac{1}{2h_1\Delta}\{(h_1+\beta)[(h_1+\lambda h_2)e^{-h_1|x-\xi|} + (h_1-\lambda h_2)e^{h_1(x+\xi)}]e^{2h_1 a} \\ &\qquad + (h_1-\beta)[(h_1+\lambda h_2)e^{-h_1(x+\xi)} + (h_1-\lambda h_2)e^{h_1|x-\xi|}]\}, \\ g_{12}^n(x,\xi) &= \frac{\lambda}{\Delta}[(h_1+\beta)e^{h_1(2a+x)} + (h_1-\beta)e^{-h_1 x}]e^{-h_2\xi}, \\ g_{21}^n(x,\xi) &= \frac{1}{\Delta}[(h_1-\beta)e^{-h_1\xi} + (h_1+\beta)e^{h_1(2a+\xi)}]e^{-h_2 x}, \end{aligned}$$

and

$$\begin{aligned} g_{22}^n(x,\xi) &= \frac{1}{2h_2\Delta}\{(h_1+\beta)[(h_1+\lambda h_2)e^{-h_2|x-\xi|} - (h_1-\lambda h_2)e^{-h_2(x+\xi)}]e^{2h_1 a} \\ &\qquad - (h_1-\beta)(h_1-\lambda h_2)[e^{-h_2(x+\xi)} + e^{-h_2|x-\xi|}]\}. \end{aligned}$$

Note that the variables x and ξ in $g_{11}^n(x,\xi)$ range between $-a \le x, \xi \le 0$, while for $g_{12}^n(x,\xi)$ we have $-a \le x \le 0 \le \xi < \infty$. For $g_{22}^n(x,\xi)$ the variables range between $0 \le x, \xi < \infty$, whilst for $g_{21}^n(x,\xi)$ we have $-a \le \xi \le 0 \le x < \infty$.

Recall that, in accordance with the fundamental rule for Fourier coefficients (Fourier–Euler formulas), the coefficients $f_{i,n}(x)$ of the series in (6.35) can be written as

$$f_{i,n}(\xi) = \frac{2}{b}\int_0^b f_i(\xi,\eta)\sin\nu\eta\, d\eta, \quad i = 1,2,\ n = 1,2,3,\ldots.$$

Upon substituting $f_{1,n}(\xi)$ and $f_{2,n}(\xi)$ in (6.48) and (6.49), and subsequently substituting $u_{1,n}(x)$ and $u_{2,n}(x)$ into the series in (6.34), we finally obtain the solution of the boundary-value problem of (6.30)–(6.33) in integral form

$$u_1(x,y) = \int_0^b \int_{-a}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{11}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$
$$+ \int_0^b \int_0^\infty \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{12}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_1,$$

and

$$u_2(x,y) = \int_0^b \int_{-a}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{21}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$
$$+ \int_0^b \int_0^\infty \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{22}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_2.$$

Thus, in compliance with the definition given in the introductory section of this chapter, we conclude that the series

$$G_{ij}(x,y;\xi,\eta) = \frac{2}{b} \sum_{n=1}^{\infty} g_{ij}^n(x,\xi) \sin \nu y \sin \nu \eta \tag{6.50}$$
$$= \frac{1}{b} \sum_{n=1}^{\infty} g_{ij}^n(x,\xi) \left[\cos \nu(y-\eta) - \cos \nu(y+\eta) \right], \quad i,j = 1,2,$$

represents the elements $G_{ij}(x,y;\xi,\eta)$ of the matrix of Green's type $\mathbf{G}(x,y;\xi,\eta)$ for the homogeneous boundary-value problem corresponding to (6.30)–(6.33).

If we set $k_1 = 0$ and $k_2 = 0$ in (6.30) we obtain $h_1 = h_2 = \nu$. This transforms the coefficients $g_{ij}^n(x,\xi)$ of the series in (6.50) to

$$g_{11}^n(x,\xi) = \frac{1}{2\nu\Delta^*}[(\nu+\beta)((1+\lambda)e^{-\nu|x-\xi|} + (1-\lambda)e^{\nu(x+\xi)})e^{2\nu a}$$
$$+ (\nu-\beta)((1+\lambda)e^{-\nu(x+\xi)} + (1-\lambda)e^{\nu|x-\xi|})],$$
$$g_{12}^n(x,\xi) = \frac{\lambda}{\nu\Delta^*}[(\nu+\beta)e^{\nu(2a+x)} + (\nu-\beta)e^{-\nu x}]e^{-\nu\xi},$$
$$g_{21}^n(x,\xi) = \frac{1}{\nu\Delta^*}[(\nu-\beta)e^{-\nu\xi} + (\nu+\beta)e^{\nu(2a+\xi)}]e^{-\nu x},$$

and

$$g_{22}^n(x,\xi) = \frac{1}{2\nu\Delta^*}[(\nu+\beta)((1+\lambda)e^{-\nu|x-\xi|} - (1-\lambda)e^{-\nu(x+\xi)})e^{2\nu a}$$
$$+ \nu(\nu-\beta)(1-\lambda)(e^{-\nu(x+\xi)} + e^{-\nu|x-\xi|})]$$

with $\Delta^* = (1+\lambda)(\nu+\beta)e^{2\nu a} - (1-\lambda)(\nu-\beta)$.

Upon close analysis, it is revealed that the expansions in (6.50), with the coefficients $g_{ij}^n(x,\xi)$, as written above, represent a series of the type

$$\sum_{n=1}^{\infty} \frac{q^n}{n} \cos n\alpha, \quad \text{where } q \leq 1 \text{ and } 0 \leq \alpha < 2\pi,$$

which diverges for $q = 1$ and $\alpha = 0$. Hence, the series in (6.50) converges non-uniformly (contains the logarithmic singularity) for the elements $G_{11}(x, y; \xi, \eta)$ and $G_{22}(x, y; \xi, \eta)$.

In order to get an idea of the convergence of the series in (6.50), we have conducted a numerical experiment. The accuracy level that we can attain by truncating the series can be observed in Figure 6.2, where we depict a profile of the elements $G_{11}(x, y; \xi, \eta)$ and $G_{21}(x, y; \xi, \eta)$, with the series (6.50) truncated to its 10th partial sum. We have chosen $a = \pi$, $b = \pi$, $\lambda = 0.01$, and $\beta = 2.5$ and the source point is placed at $(-1.0, 1.5) \in \Omega_1$.

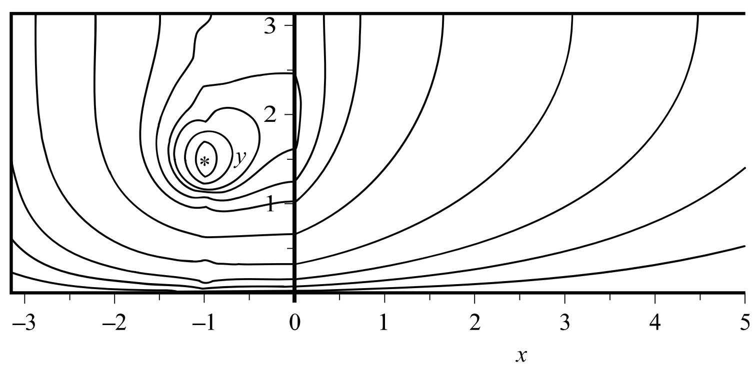

Figure 6.2. Convergence of the series in (6.50).

Based on the resulting numerical data, we can conclude that, if we require accurate values of $\mathbf{G}(x, y; \xi, \eta)$ in the entire region Ω, then a series-only representation of similar to (6.50) is not efficient and cannot be recommended. To enhance its computability, we will implement the approach originally proposed in [43]. In doing so, the coefficients $g_{11}^n(x,\xi)$ of $G_{11}(x, y; \xi, \eta)$ are transformed as

$$\begin{aligned} g_{11}^n(x,\xi) &= \frac{p(x,\xi)}{2\nu}\left[\frac{1}{\Delta^*} - \frac{e^{-2\nu a}}{\nu(1+\lambda)} + \frac{e^{-2\nu a}}{\nu(1+\lambda)}\right] \\ &= \frac{p(x,\xi)}{2\nu}\left[\frac{(1-\lambda)(\nu-\beta)e^{-2\nu a} - \beta(1+\lambda)}{\nu(1+\lambda)\Delta^*} + \frac{e^{-2\nu a}}{\nu(1+\lambda)}\right], \end{aligned}$$

where

$$p(x,\xi) = \nu p_1(x,\xi) + \beta p_2(x,\xi)$$

with

$$p_1(x,\xi) = [((1+\lambda)e^{-\nu|x-\xi|} + (1-\lambda)e^{\nu(x+\xi)})e^{2\nu a} + ((1+\lambda)e^{-\nu(x+\xi)} + (1-\lambda)e^{-\nu|x-\xi|})] \tag{6.51}$$

and

$$p_2(x,\xi) = [((1+\lambda)e^{-\nu|x-\xi|} + (1-\lambda)e^{\nu(x+\xi)})e^{2\nu a} - ((1+\lambda)e^{-\nu(x+\xi)} + (1-\lambda)e^{-\nu|x-\xi|})].$$

This splits the series representation of $G_{11}(x,y;\xi,\eta)$ into two segments, the first of which

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p(x,\xi)\left[(1-\lambda)(\nu-\beta)e^{-2\nu a} - \beta(1+\lambda)\right]}{\nu^2\left((1+\lambda)(\nu+\beta)e^{2\nu a} - (1-\lambda)(\nu-\beta)\right)} \times [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \tag{6.52}$$

converges uniformly at the rate of

$$\sum_{n=1}^{\infty} \frac{q^n}{n^2}\cos n\alpha, \quad \text{where } q \le 1 \text{ and } 0 \le \alpha < 2\pi.$$

Hence it is already in a computer-friendly form. With regard to the second series in $G_{11}(x,y;\xi,\eta)$, written as

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p(x,\xi)}{\nu^2 e^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)]$$

we can split it into

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p_1(x,\xi)}{\nu e^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] + \frac{\beta}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p_2(x,\xi)}{\nu^2 e^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)], \tag{6.53}$$

where the second expansion converges uniformly at the same rate as (6.52). Hence, by combining the two, we obtain a uniformly convergent series component for $G_{11}(x,y;\xi,\eta)$ as

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{(1-\lambda)(\nu-\beta)p_1(x,\xi)e^{-2\nu a} - \beta(1+\lambda)p_3(x,\xi)}{\nu\left((1+\lambda)(\nu+\beta)e^{2\nu a} - (1-\lambda)(\nu-\beta)\right)} \times [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \tag{6.54}$$

where

$$p_3(x,\xi) = 2((1+\beta)\mathrm{e}^{-\nu(x+\xi)} + (1-\beta)\mathrm{e}^{\nu|x-\xi|}).$$

The non-uniform convergence of the first series in (6.53)

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p_1(x,\xi)}{\nu \mathrm{e}^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \tag{6.55}$$

is evident. The good news is, however, that it is completely summable. To prepare for the summation, we substitute $p_1(x,\xi)$ from (6.51) in (6.55), and rewrite the latter as

$$\begin{aligned} \frac{1}{2b(1+\lambda)} \Bigg\{ (1+\beta) \Bigg[& \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\nu|x-\xi|}}{\nu} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \\ & + \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\nu(2a+x+\xi)}}{\nu} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \Bigg] \\ + (1-\beta) \Bigg[& \sum_{n=1}^{\infty} \frac{\mathrm{e}^{\nu(x+\xi)}}{\nu} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \\ & + \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\nu(2a-|x-\xi|)}}{\nu} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \Bigg] \Bigg\}. \end{aligned} \tag{6.56}$$

Recalling the expression for $\nu = (2n-1)\pi/2b$, we sum all of the above series using the standard summation formula [1, 27]

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{2n-1} \cos(2n-1)\alpha = \frac{1}{4} \ln \frac{1+2q\cos\alpha+q^2}{1-2q\cos\alpha+q^2}$$

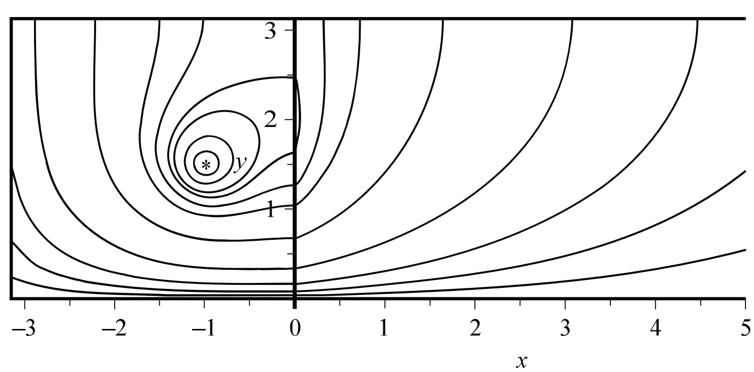

Figure 6.3. Improvement of the convergence.

yielding the following closed form for (6.56)

$$
\begin{aligned}
\frac{1}{2\pi(1+\lambda)}\Bigg\{&(1+\beta)\ln\Bigg(\frac{1+2e^{\omega(x-\xi)}\cos(y-\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos(y-\eta)+e^{2\omega(x-\xi)}}\\
&\times\frac{1-2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}{1+2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}\\
&\times\frac{1+2e^{\omega(2a+x+\xi)}\cos(y-\eta)+e^{2\omega(2a+x+\xi)}}{1-2e^{\omega(2a+x+\xi)}\cos(y-\eta)+e^{2\omega(2a+x+\xi)}}\\
&\times\frac{1-2e^{\omega(2a+x+\xi)}\cos(y+\eta)+e^{2\omega(2a+x+\xi)}}{1+2e^{\omega(2a+x+\xi)}\cos(y+\eta)+e^{2\omega(2a+x+\xi)}}\Bigg)\\
&+(1-\beta)\ln\Bigg(\frac{1+2e^{\omega(x+\xi)}\cos(y-\eta)+e^{2\omega(x+\xi)}}{1-2e^{\omega(x+\xi)}\cos(y-\eta)+e^{2\omega(x+\xi)}}\\
&\times\frac{1-2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}{1+2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}\\
&\times\frac{1+2e^{\omega(2a-x+\xi)}\cos(y-\eta)+e^{2\omega(2a-x+\xi)}}{1-2e^{\omega(2a-x+\xi)}\cos(y-\eta)+e^{2\omega(2a-x+\xi)}}\\
&\times\frac{1-2e^{\omega(2a-x+\xi)}\cos(y+\eta)+e^{2\omega(2a-x+\xi)}}{1+2e^{\omega(2a-x+\xi)}\cos(y+\eta)+e^{2\omega(2a-x+\xi)}}\Bigg)\Bigg\},
\end{aligned}
\tag{6.57}
$$

where $\omega = \pi/2b$.

Hence, the sum of the uniformly convergent series in (6.54) and the expression in (6.57) provide a computer-friendly formula of $G_{11}(x, y; \xi, \eta)$ of $\mathbf{G}(x, y; \xi, \eta)$. The improvement of the series convergence for the rest of the elements can be accomplished in a similar manner.

To illustrate the accuracy improvement attained by the development we have just described, with regard to the convergence of the series representing elements of the matrix of Green's type, we depict, in Figure 6.3, the same profile of $\mathbf{G}(x, y; \xi, \eta)$ as in Figure 6.2. We employed the computer-friendly elements of $\mathbf{G}(x, y; \xi, \eta)$, with their series components truncated to the 10th partial sum.

Example 6.3. Consider yet another problem where its matrix of Green's type can be obtained in closed series-free form. Let the half-plane $\Omega_2 = \{a < r < \infty, 0 < \varphi < \pi\}$, reduced by a semi-circular cut-out of radius a, be filled with a conducting (λ_2) isotropic homogeneous material. Let also Ω_2 contain a semi-circular inclusion $\Omega_1 = \{0 < r < a, 0 < \varphi < \pi\}$ made out of a foreign conducting (λ_1) isotropic homogeneous material. To determine the potential field generated in $\Omega = \Omega_1 \cup \Omega_2$ by